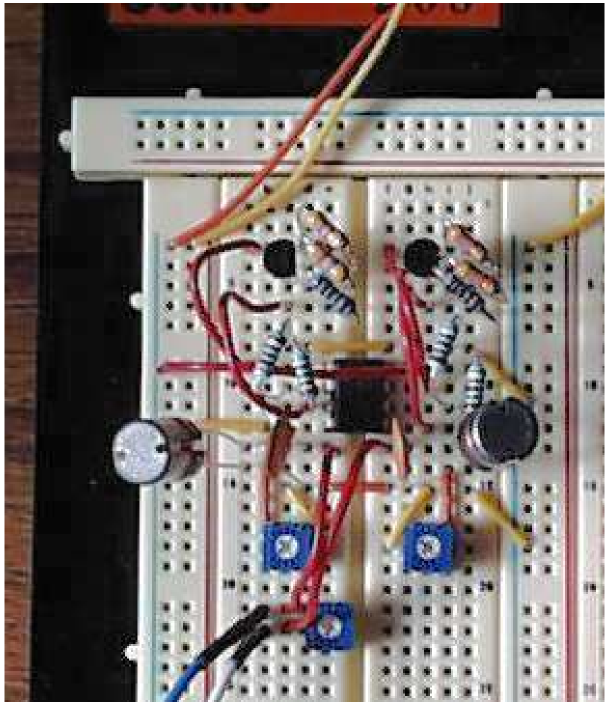
Bifurcation and Chaos in Coupled BVP Oscillators



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History of BVP Oscillator

- Introduced as a simplified model of Hodkin-Huxley equation (four-dimensional autonomous system)
- alias is FitzHugh-Nagumo oscillator
- extraction of excitatory activity from HH equation

$$
\begin{aligned}
\dot{x} &= c(x - \frac{x^3}{3} + y + z) \\
\dot{y} &= -\frac{1}{c}(x + by - a)
\end{aligned}
$$

Analogue of BVP equation

An natural extension of van der Pol equation.

- 2nd dimensional autonomous system.
- Evaluate a resistance in a coil.
- Adding a bias source to destroy symmetric property.

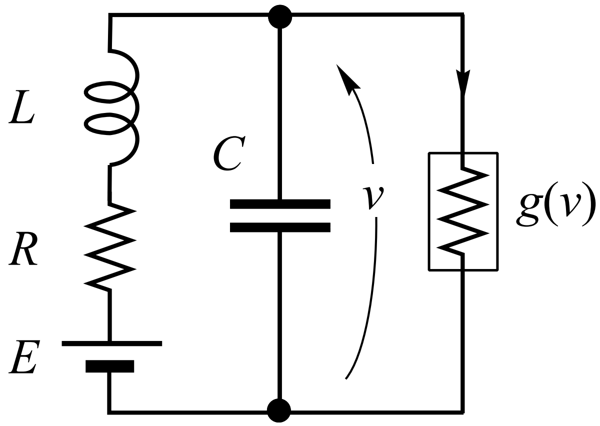
- A. N. Bautin, "Qualitative investigation of a Particular Nonlinear System," PPM, vol. 39, No. 4, pp. 606–615, 1975. → Topological classification of solutions in BVP equation
- Doi, et. al: Response of BVP with an impulsive force

Coupled BVP equation

- O. Papy, H. Kawakami: Analysis on coupled BVP equations from symmetry point of view.
- Kitajima: Chaos generation from symmetry coupled BVP equations

No chaotic oscillations in symmetrical configuration of coupled BVP equations

BVP Oscillator

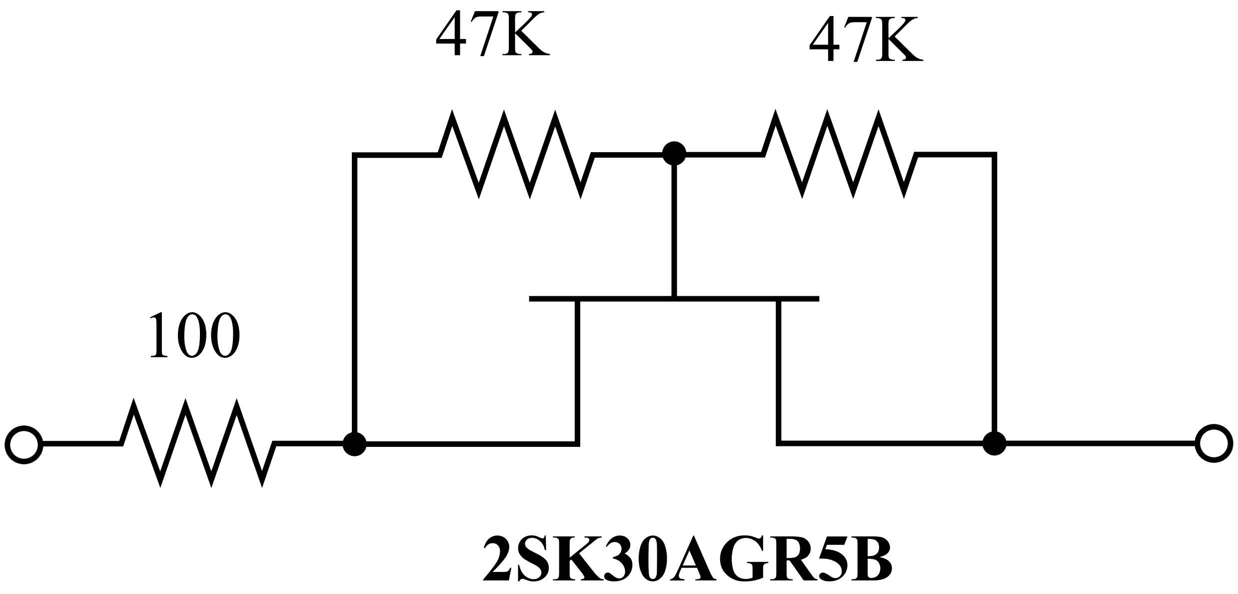


Circuit equation:

$$
\begin{aligned}
C\frac{dv}{dt} &= -i - g(v) \\
L\frac{di}{dt} &= v - ri + E
\end{aligned}
\qquad (1)
$$

Nonlinear conductor

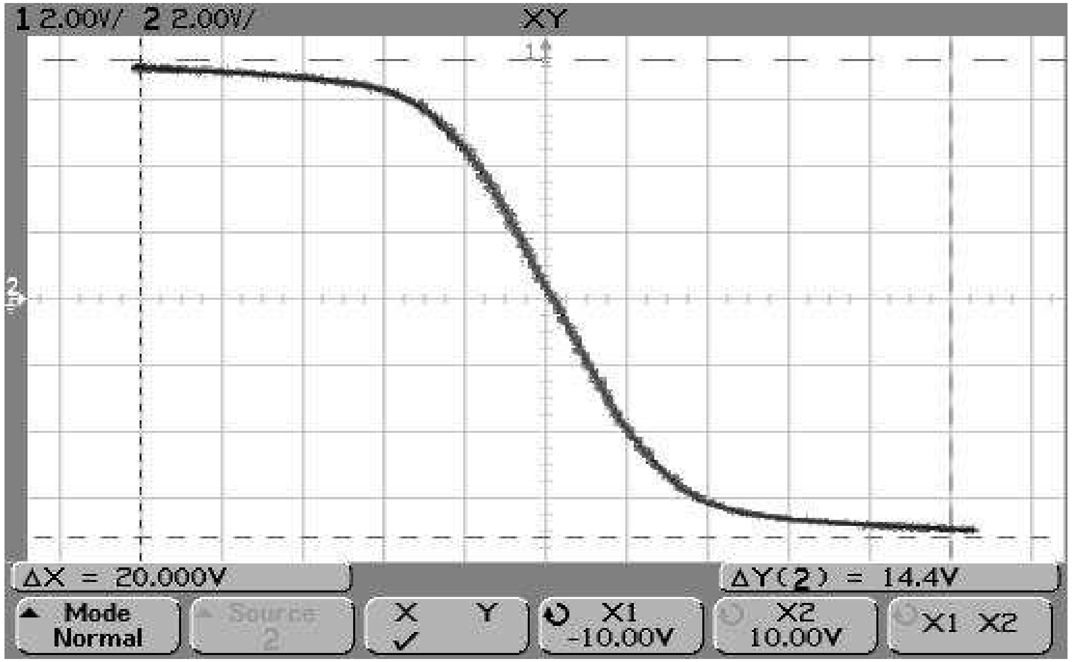
2SK30A FET:



$$g(v) = -a \tanh bv$$

Fitting

Marquardt-Levenberg method (nonlinear least square method)



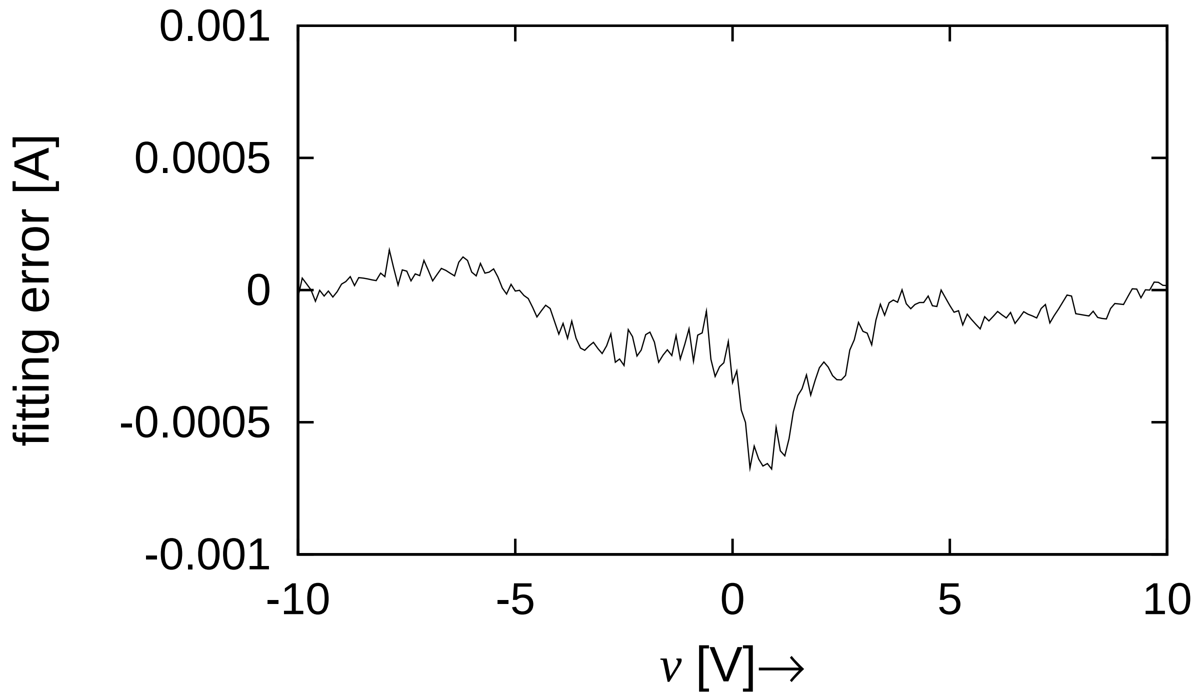
$a = 6.89099 \times 10^{-3}$, $b = 0.352356$

Approximation error

$$g(v) = -a \tanh bv$$

with $a = 6.89099 \times 10^{-3}$, $b = 0.352356$

Theoretical value versus measurement value.



A reasonable approximation.

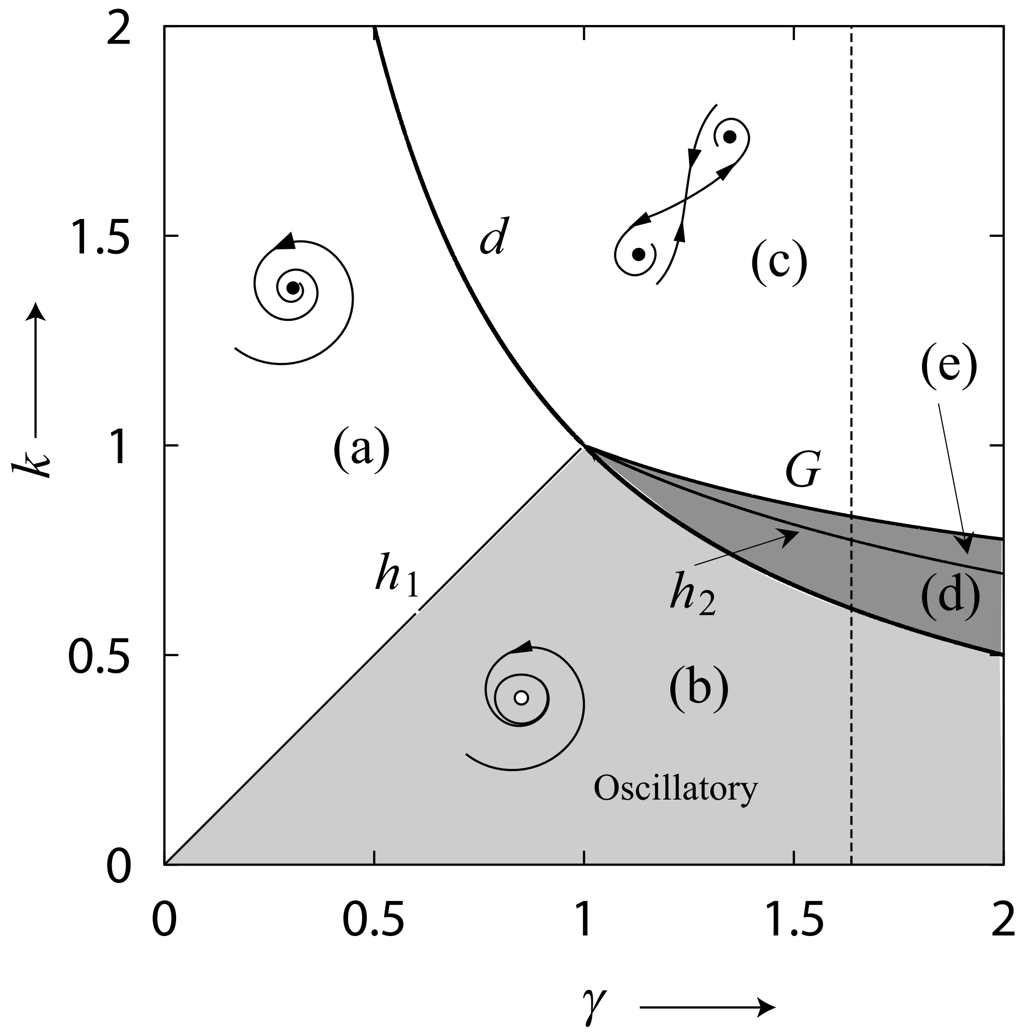
BVP equation

$$\begin{aligned} \dot{x} &= -y + \tanh \gamma x \\ \dot{y} &= x - ky. \end{aligned}$$

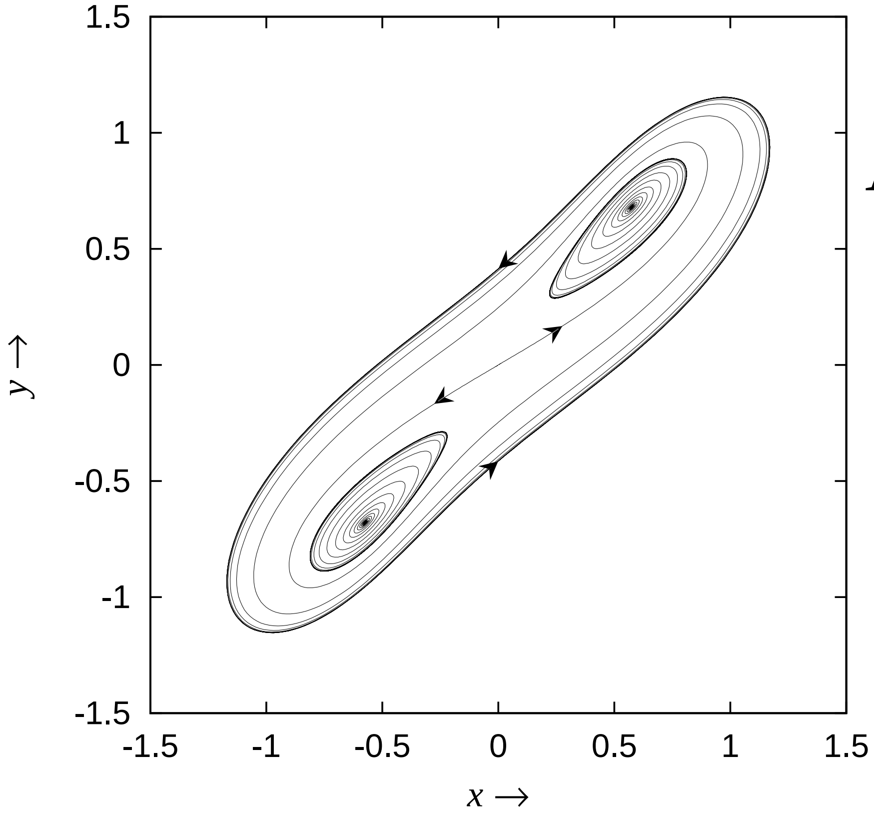
$$\cdot = d/d\tau, \quad x = \sqrt{\frac{C}{L}}v, \quad y = \frac{i}{a}$$

$$\tau = \frac{1}{\sqrt{LC}}t, \quad k = r\sqrt{\frac{C}{L}}, \quad \gamma = ab\sqrt{\frac{L}{C}}$$

Bifurcation of equilibria for single BVP oscillator



An example phase portrait



At region (d):

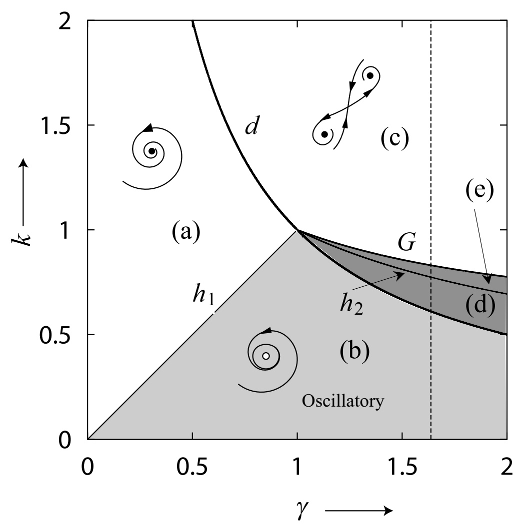
- Two stable sinks.
- A saddle.
- Two unstable limit cycles.
- A stable limit cycle.

Circuit parameters setup

$$L = 10\ [\mathrm{mH}], \quad C = 0.022\ [\mu\mathrm{F}]$$

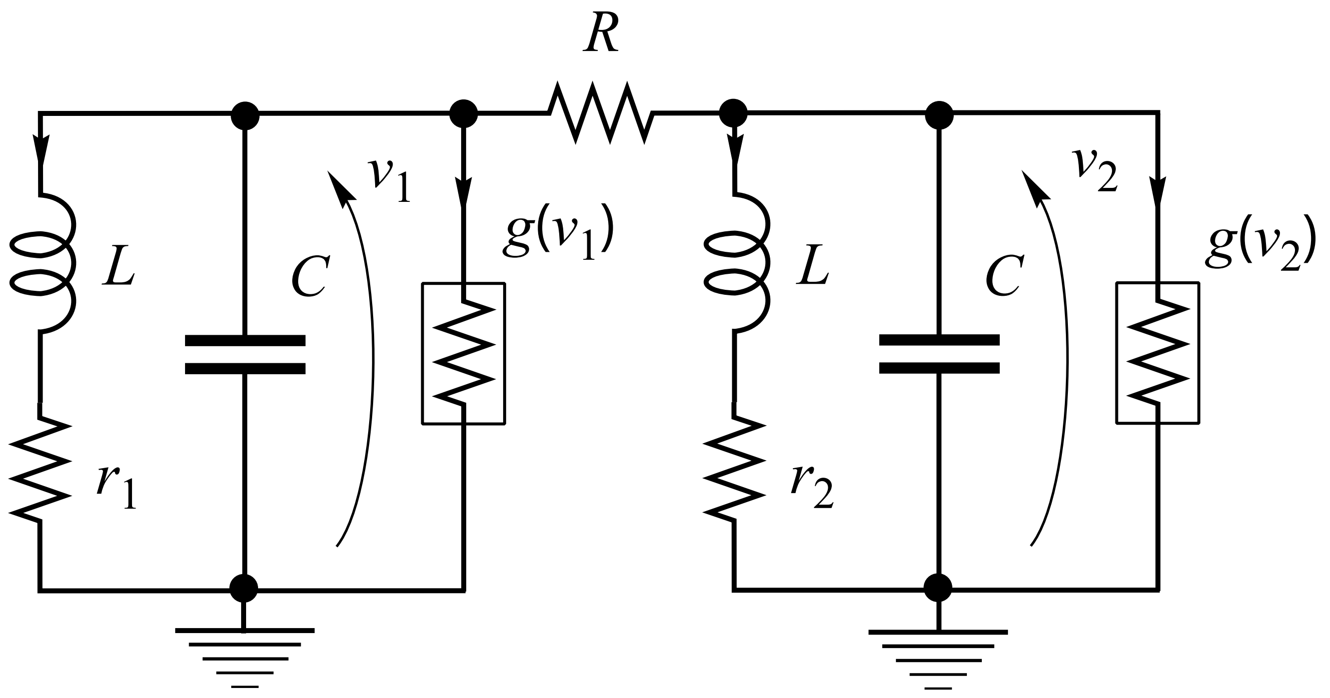
$$\Downarrow$$

$$\gamma = 1.6369909, \ \sqrt{\frac{L}{C}} = 674.19986.$$



Coupled BVP oscillators

A pair of BVP oscillators coupled by a register R.



$$G = 1/R$$

Circuit equation

$$
\begin{aligned}
C\frac{dv_1}{dt} &= -i_1 + a\tanh bv_1 - G(v_1 - v_2) \\
L\frac{di_1}{dt} &= v_1 - ri_1 \\
C\frac{dv_2}{dt} &= -i_2 + a\tanh bv_2 - G(v_2 - v_1) \\
L\frac{di_2}{dt} &= v_2 - ri_2
\end{aligned} \tag{2}
$$

Scaling

$$x_j = \sqrt{\frac{C}{L}} v_j, \quad y_j = \frac{i_j}{a}, \quad k_j = r_j \sqrt{\frac{C}{L}}, \quad j = 1, 2.$$

$$\tau = \frac{1}{\sqrt{LC}} t, \gamma = ab\sqrt{\frac{L}{C}}, \delta = \sqrt{\frac{L}{C}} G.$$

$$\begin{aligned}
\dot{x}_1 &= -y_1 + \tanh \gamma x_1 - \delta(x_1 - x_2) \\
\dot{y}_1 &= x_1 - k_1 y_1 \\
\dot{x}_2 &= -y_2 + \tanh \gamma x_2 - \delta(x_2 - x_1) \\
\dot{y}_2 &= x_2 - k_2 y_2
\end{aligned}$$

Symmetry

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

where, $\boldsymbol{f} : \boldsymbol{R}^n \to \boldsymbol{R}^n : C^\infty$ for $\boldsymbol{x} \in \boldsymbol{R}^n$.

$$P : \boldsymbol{R}^n \to \boldsymbol{R}^n$$
$$\boldsymbol{x} \mapsto P\boldsymbol{x}$$

P-invariant equation:

$$\boldsymbol{f}(P\boldsymbol{x}) = P\boldsymbol{f}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \boldsymbol{R}^n$$

A matrix P

- in case $k_1 = k_2$,

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

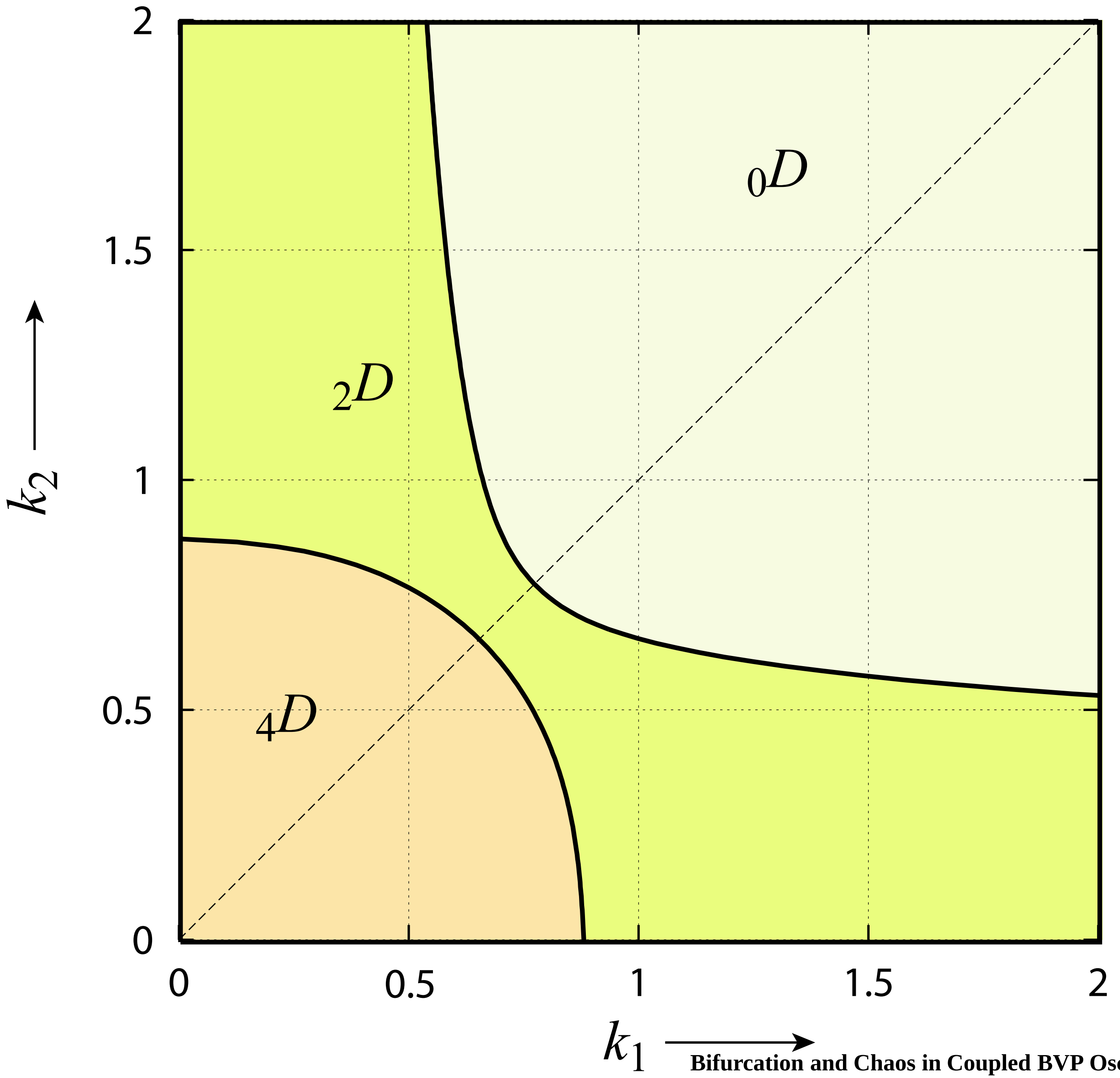
$$\Gamma = \{P, -P, I_n, -I_n\}$$

  forms a group for production.

- in case $k_1 \neq k_2$:

$$\Gamma = \{I_n, -I_n\}$$

Bifurcation of Equilibria



Poincaré mapping

A solution $\boldsymbol{\varphi}(t)$:

$$\boldsymbol{x}(t) = \boldsymbol{\varphi}(t, \boldsymbol{x}_0), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 = \boldsymbol{\varphi}(0, \boldsymbol{x}_0)\,.$$

Poincaré section:

$$\Pi = \{\, \boldsymbol{x} \in \boldsymbol{R}^n \;\mid\; q(\boldsymbol{x}) = 0 \,\}\;,$$

$$T : \hat{\Pi} \to \Pi; \quad \tilde{\boldsymbol{x}} \mapsto \boldsymbol{\varphi}(\tau(\tilde{\boldsymbol{x}}), \tilde{\boldsymbol{x}})\,,$$

The fixed point $\boldsymbol{x}_0$ for the limit cyclde $\boldsymbol{\varphi}(t)$:

$$T(\boldsymbol{x}_0) = \boldsymbol{x}_0.$$

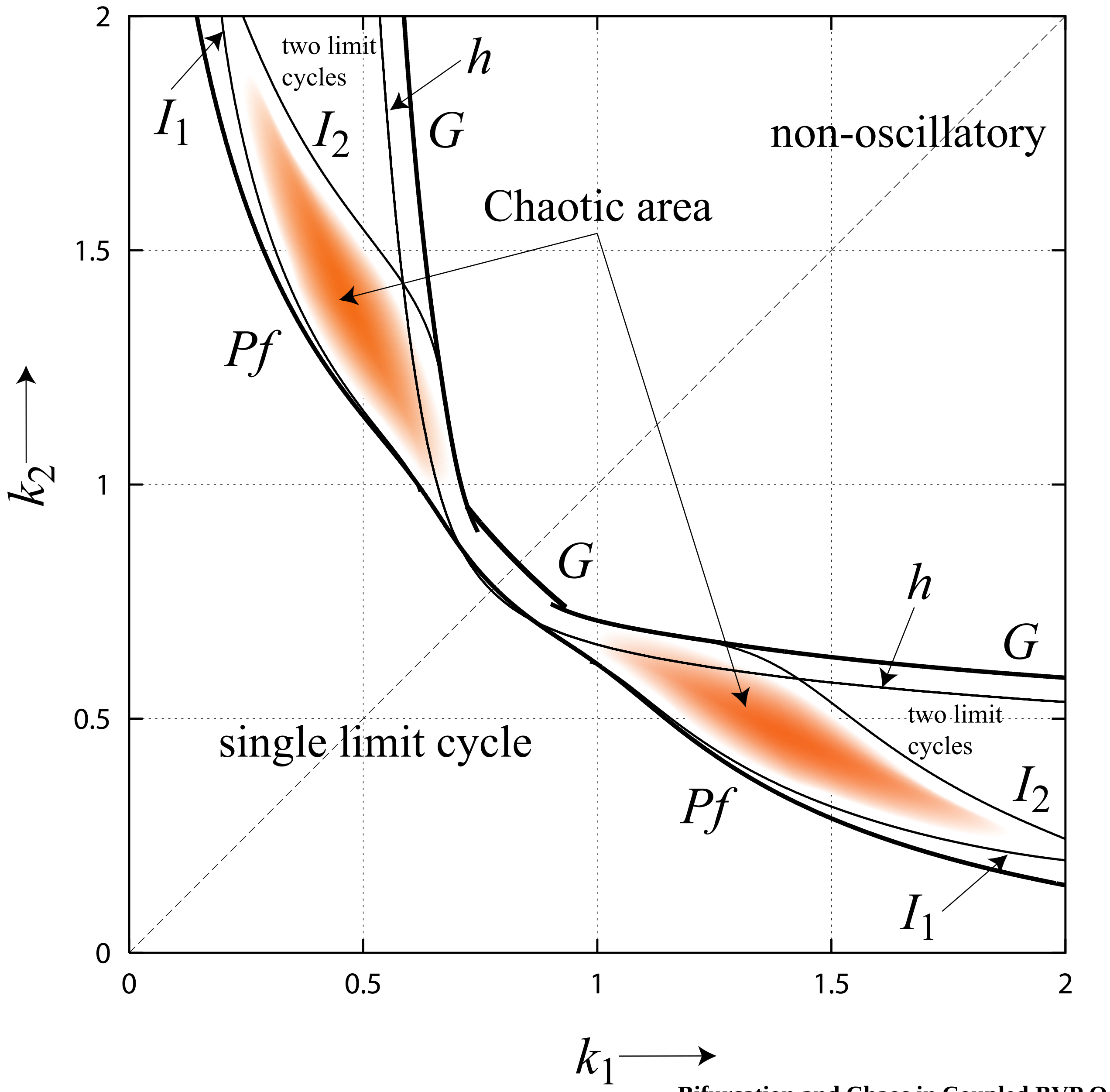
Characteristic equation

$$\chi(\mu) = \det\left(\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} - \mu I_n\right).$$

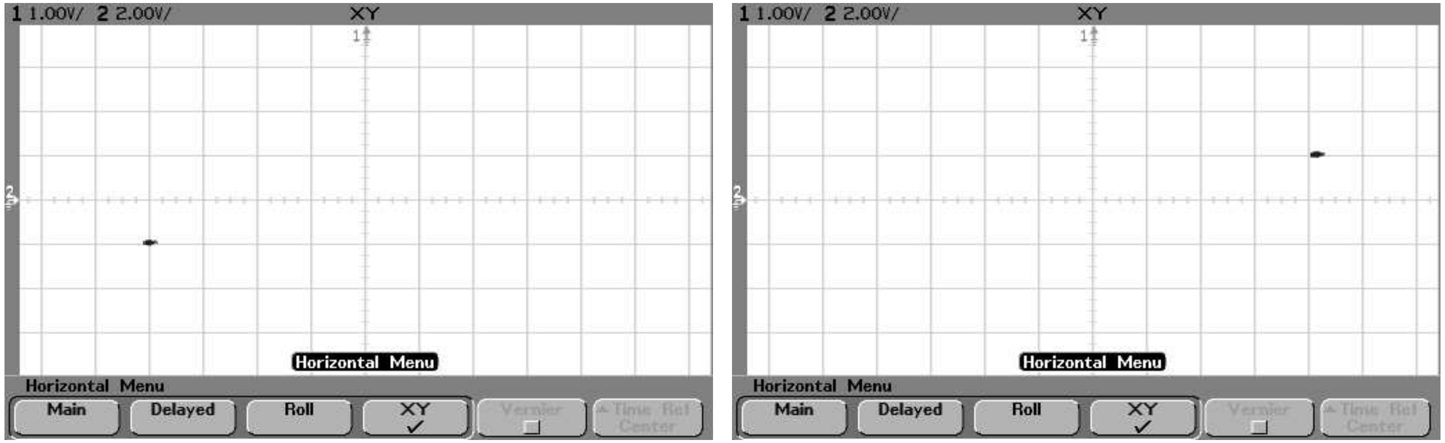
Local bifurcations:

- $\mu = 1$: tangent bifurcation G
- $\mu = -1$: period-doubling bifurcation I
- $\mu = e^{j\theta}$: Neimark-Sacker bifurcation NS
- $\mu = \pm 1$: Pitchfork bifurcation Pf

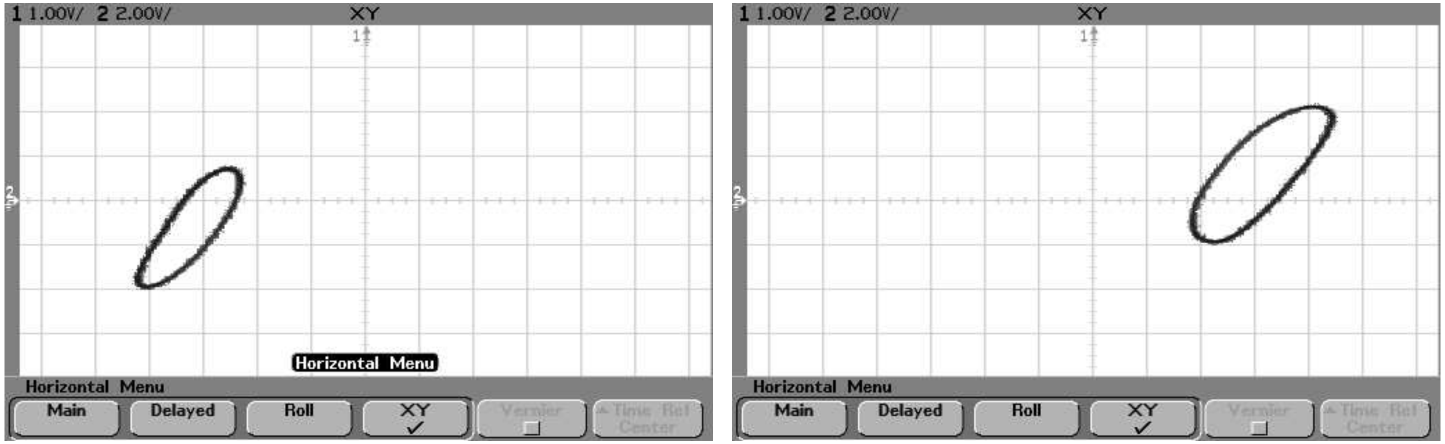
Bifurcation diagram, $\delta = 0.337 (R = 2000[\Omega])$



Hopf bifurcation $k_1 = 1.18 (r_1 \approx 800)$

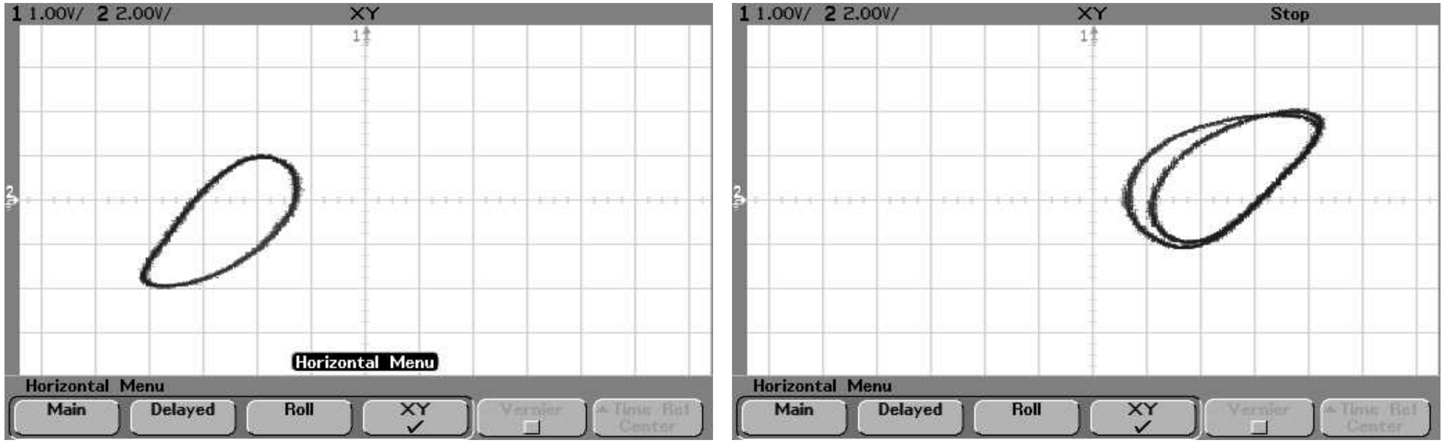


$r_2 \approx 600$

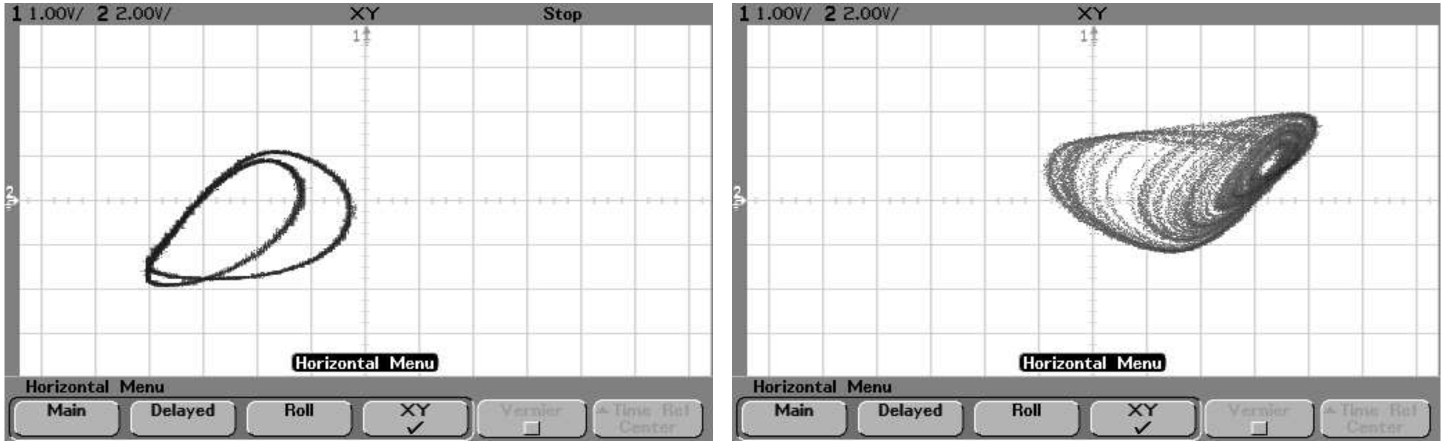


$r_2 \approx 400$

Period-doubling cascade

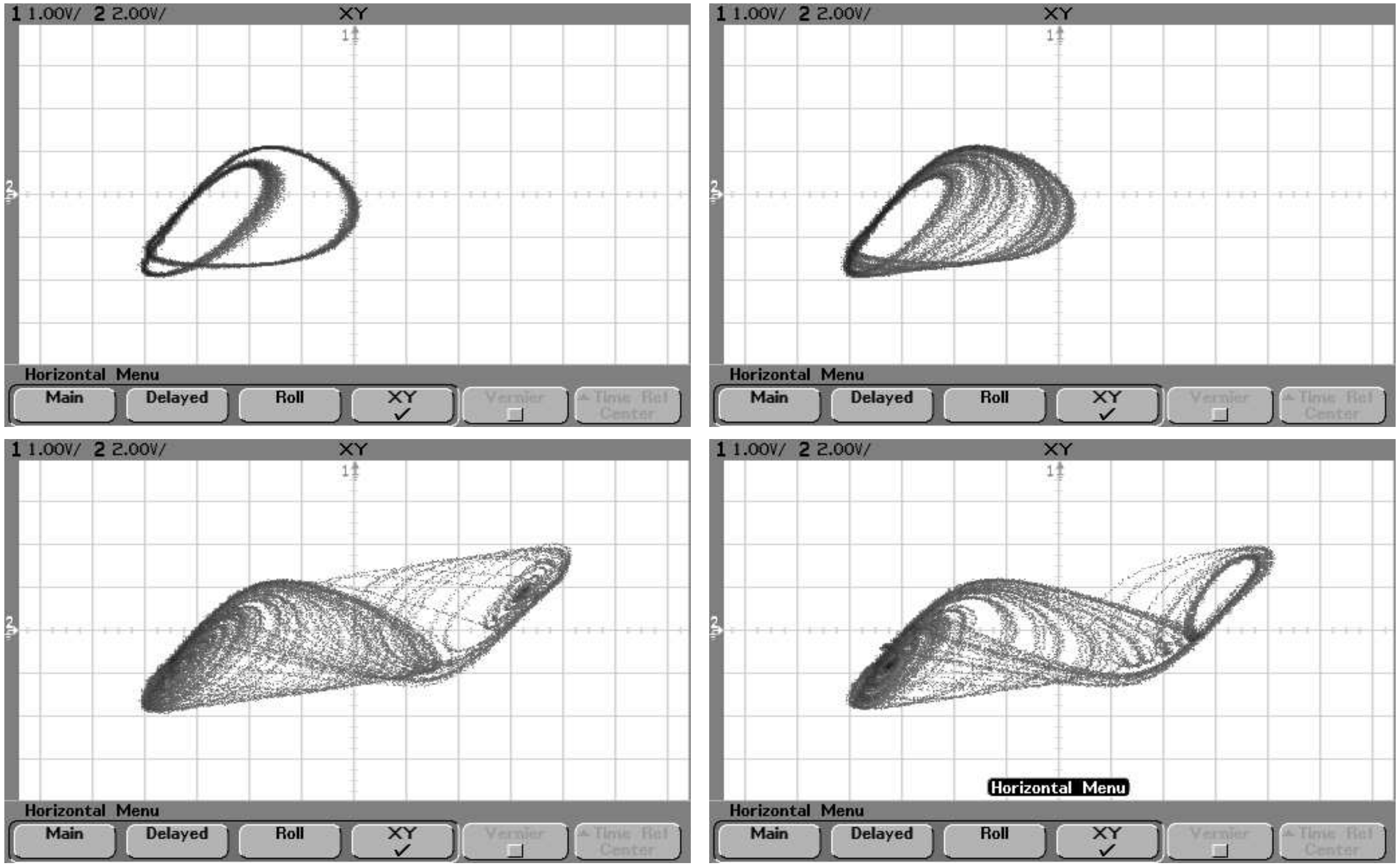


$r_2 \approx 400(k \approx 0.72)$



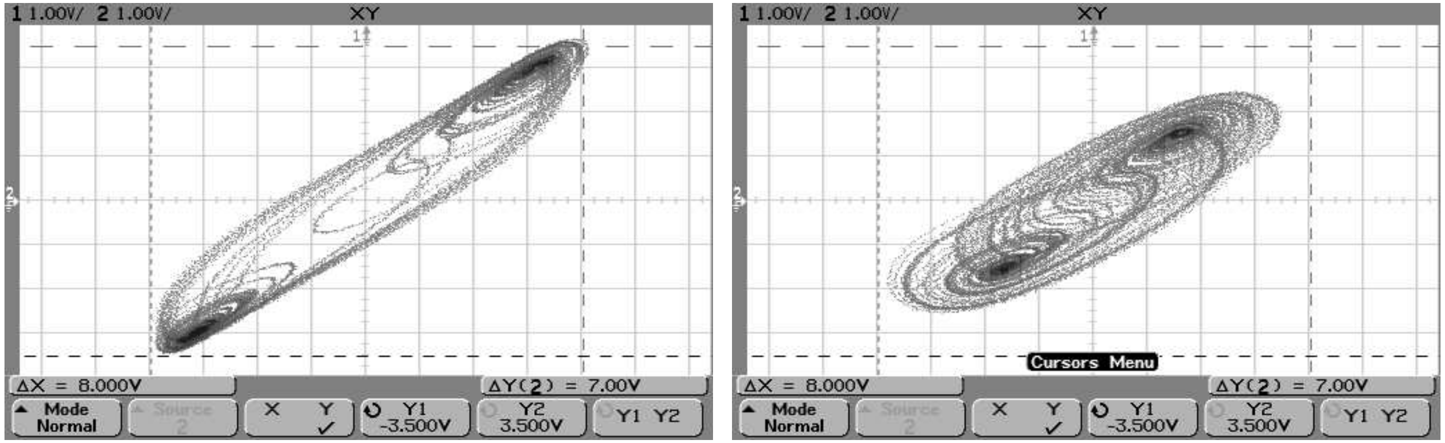
$r_2 \approx 395[\Omega]$ Rössler-type attractor

Presence of Double scroll

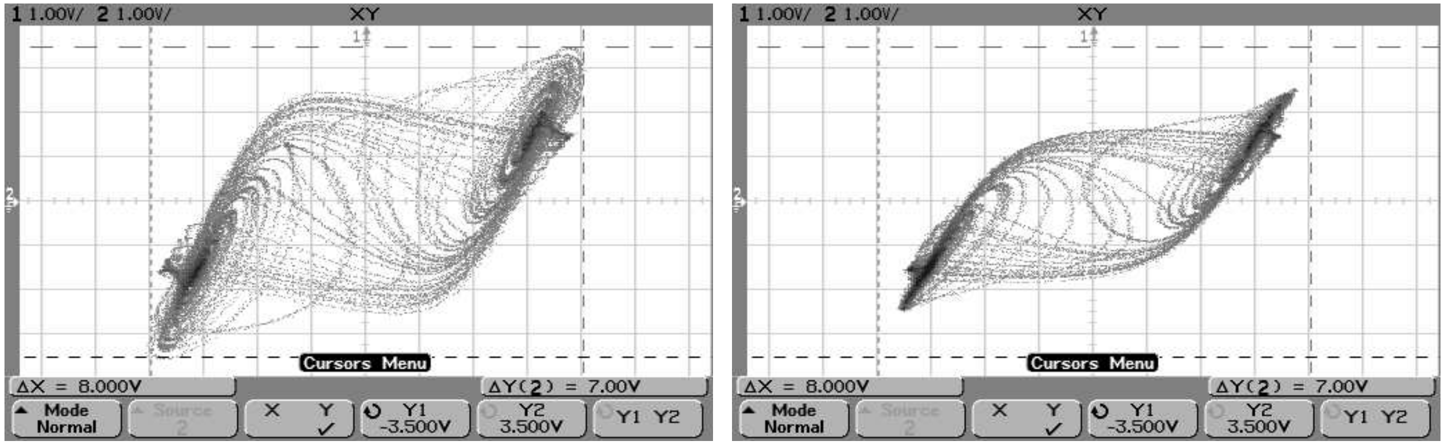


$$r_2 \approx 390 \rightarrow r_2 \approx 380[\Omega].$$

Chaotic attractor. $r_1 = 370[\Omega]$

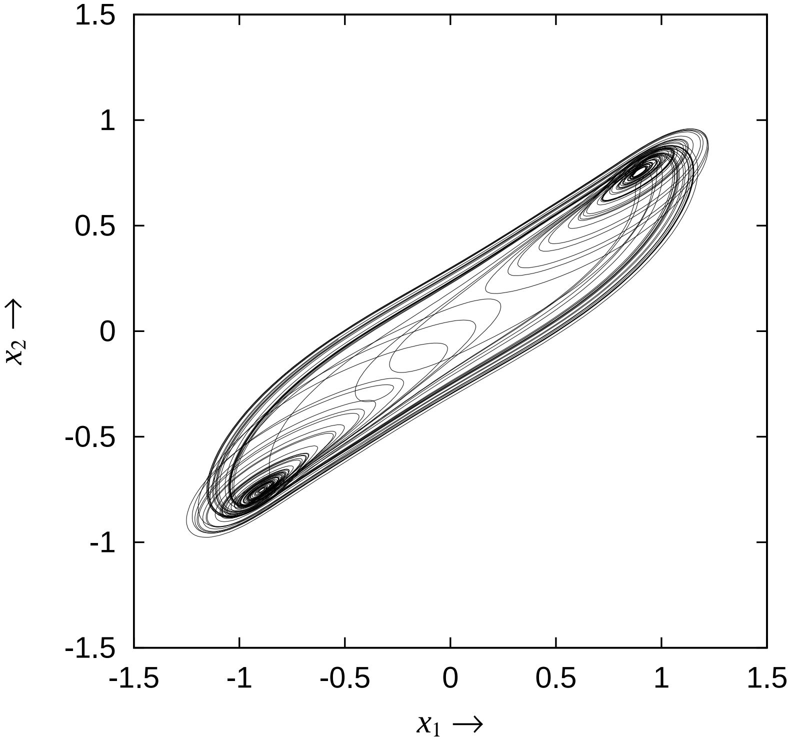


(a) v_1-r_1i_1. (b) v_2-r_2i_2.

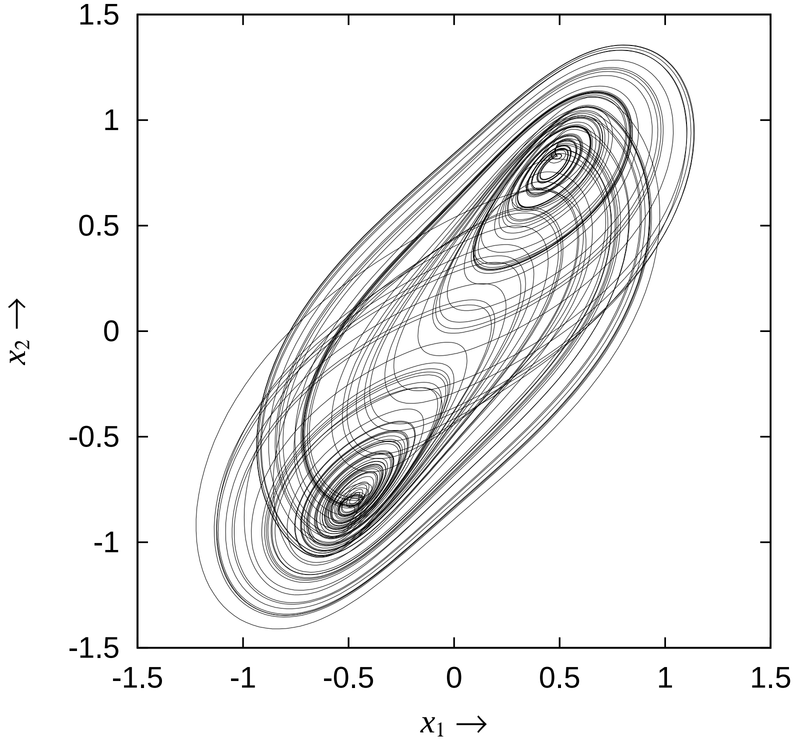


(c) v_1-v_2, (d) r_1v_1-r_2i_2.

Numerical simulation

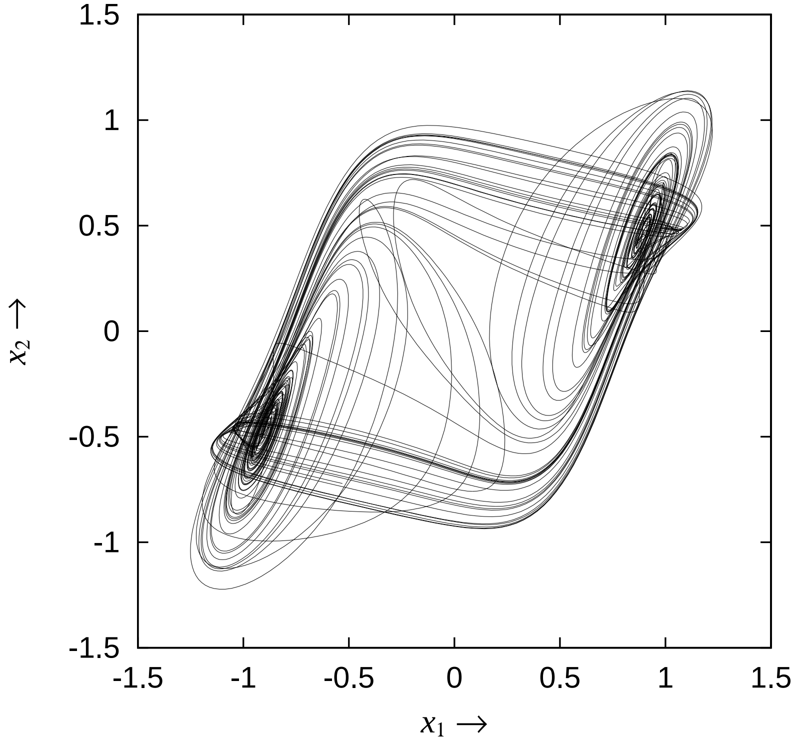


x_1-y_1

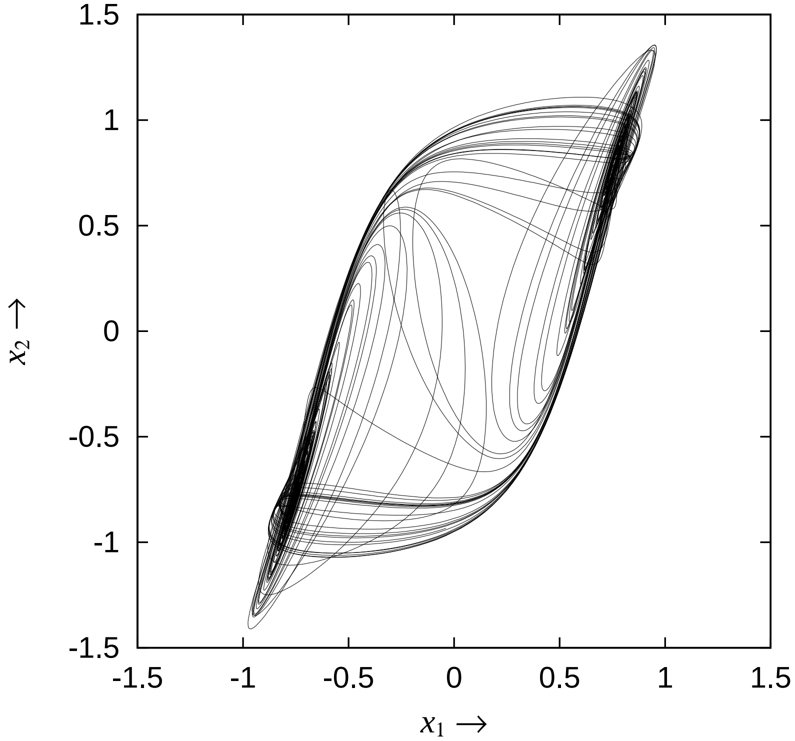


x_2-y_2

Phase portrait between oscillators



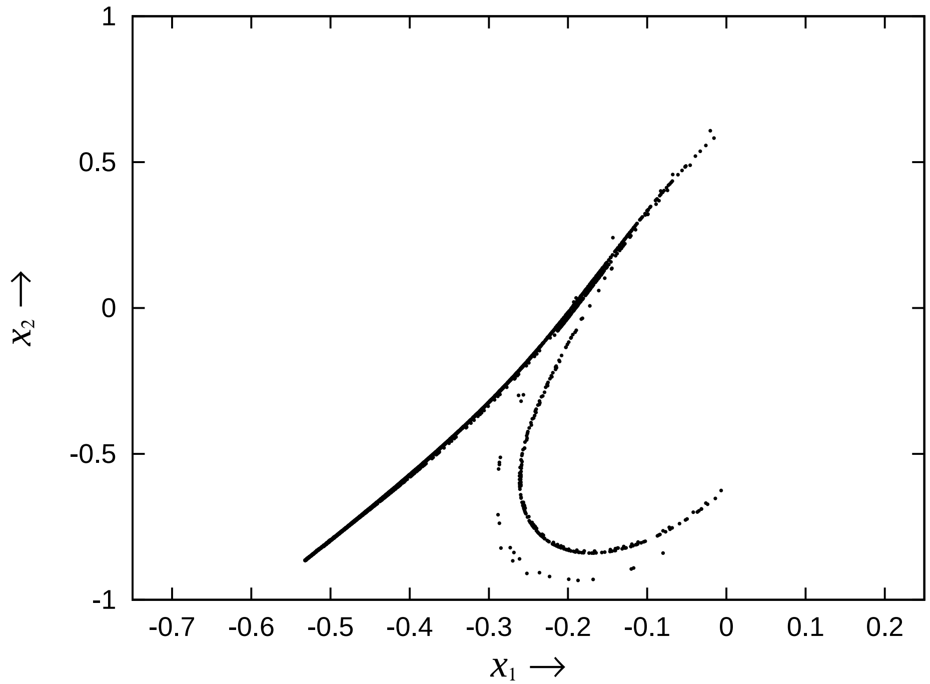
(c) x_1-x_2

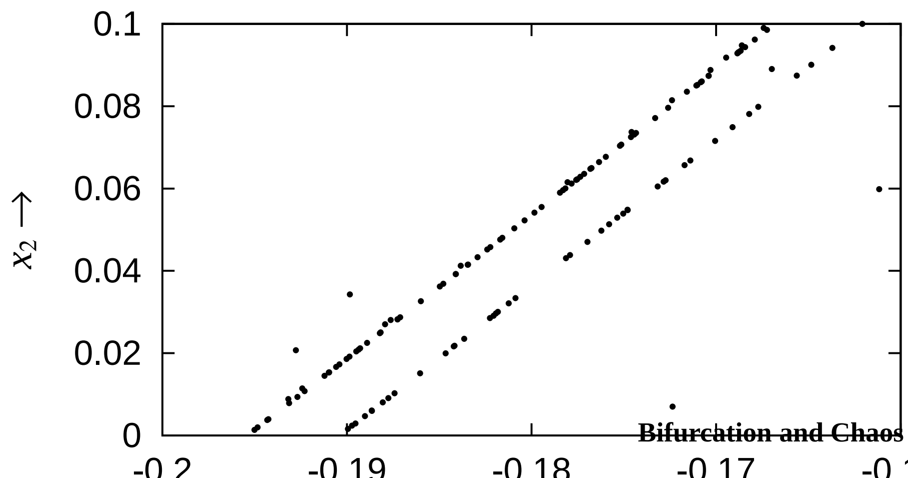


(d) y_1-y_2

Poincaré mapping

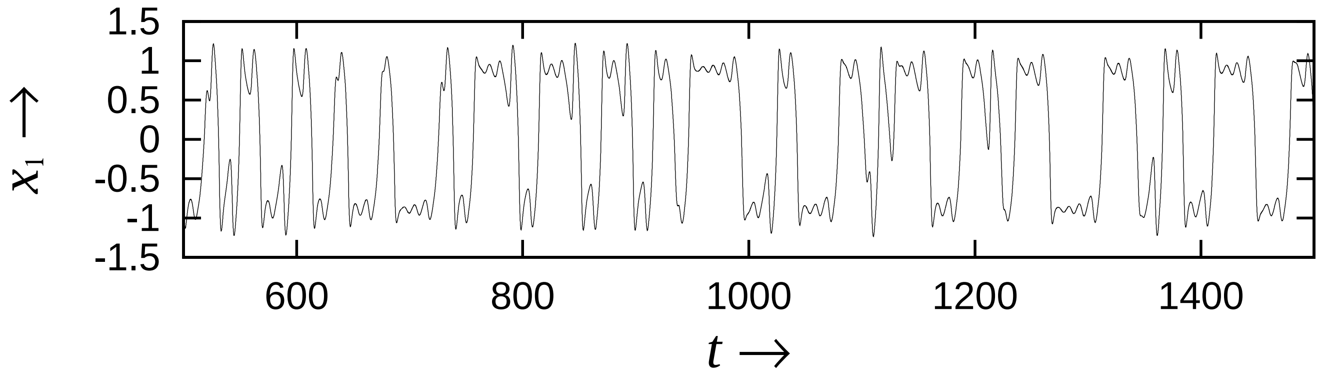
x_1-x_2 plane: $\Pi = \{\boldsymbol{x} | q(\boldsymbol{x}) = y_1 = 0\}$

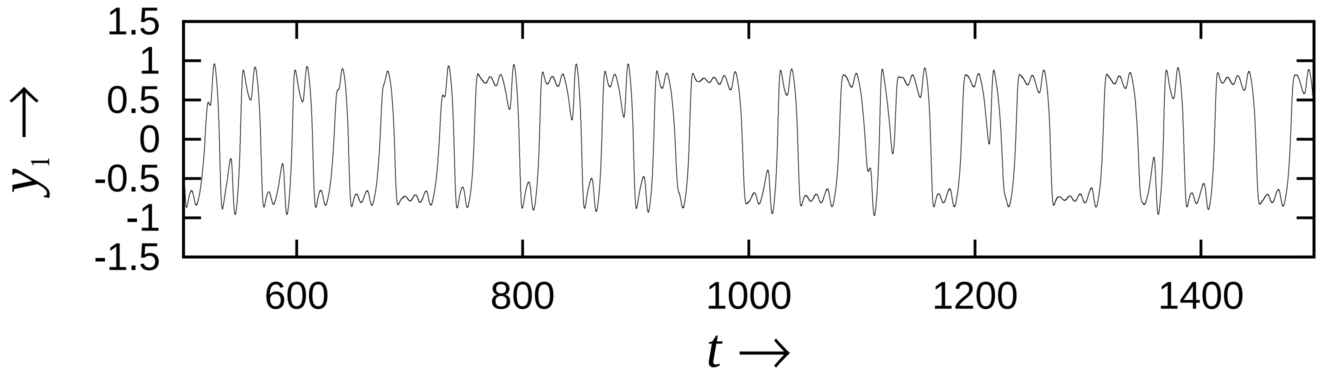




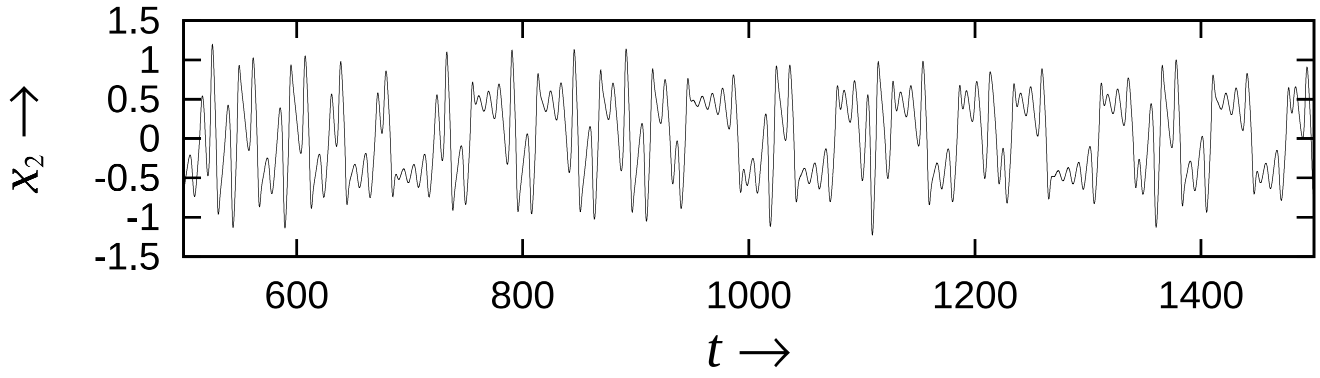
Time response of Oscillator 1

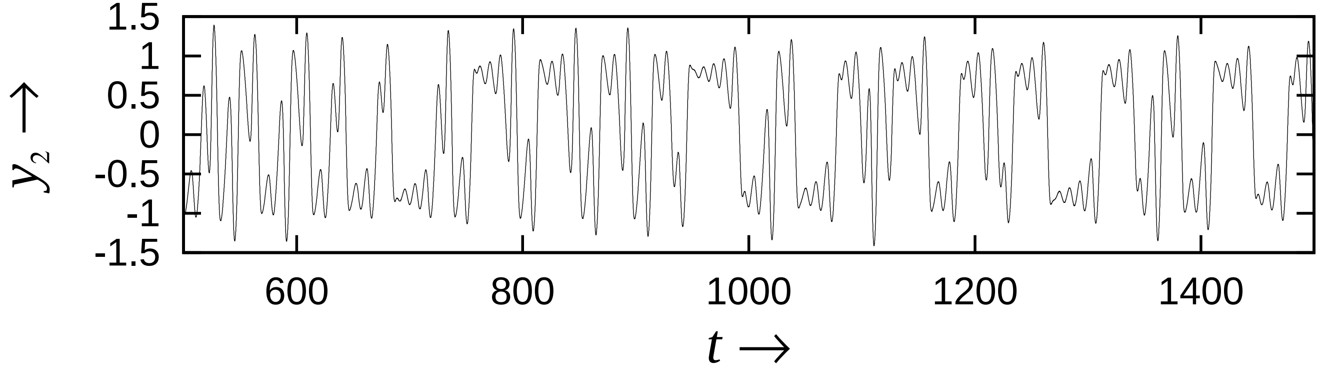
Chaotic attractor. $\delta = 0.337$, $k_1 = 1.187$, $k_2 = 0.593$, ($R = 2000[\Omega]$, $r_1 = 400[\Omega]$, $r_2 = 800[\Omega]$).



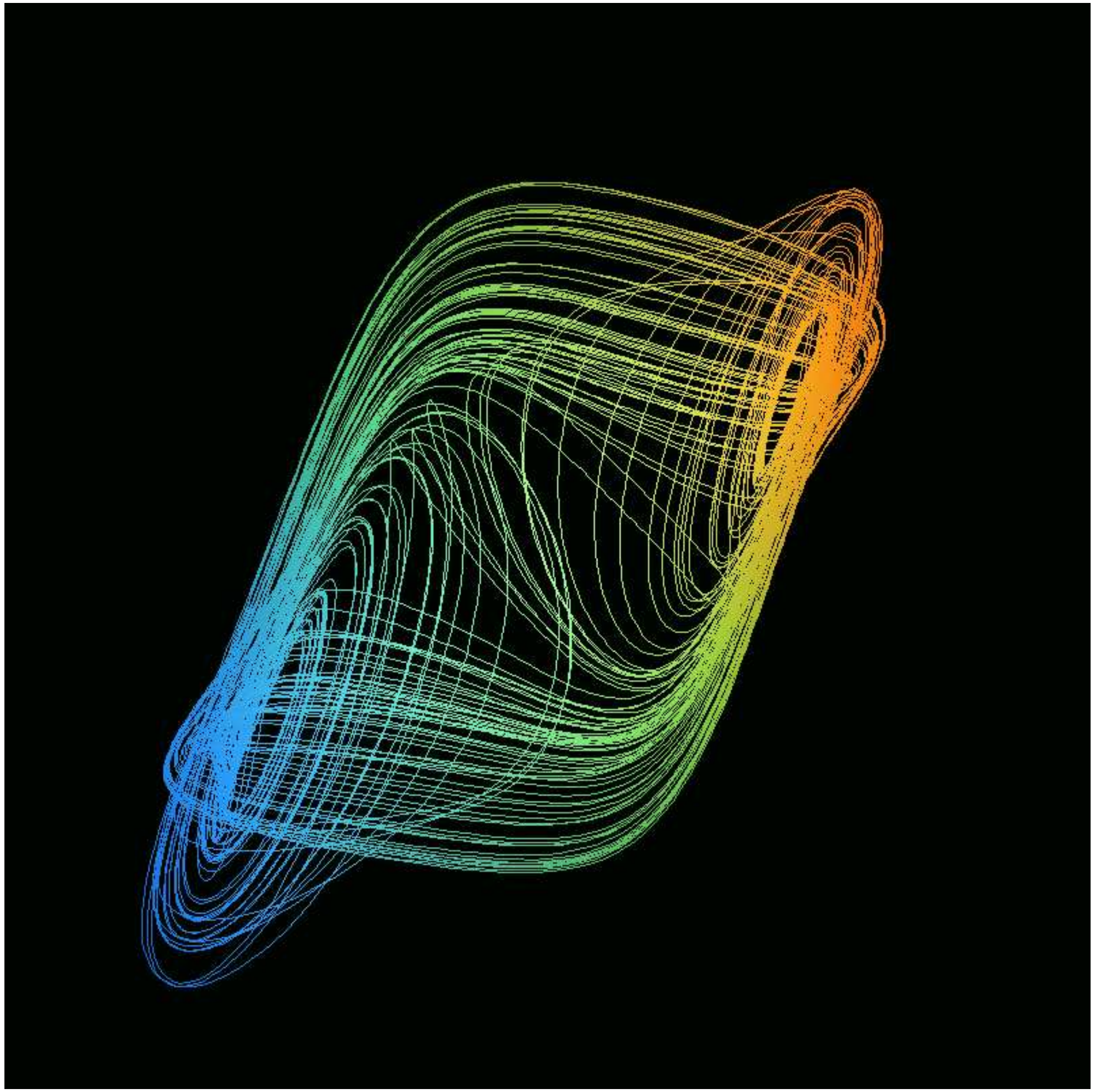


Time response of Oscillator 2

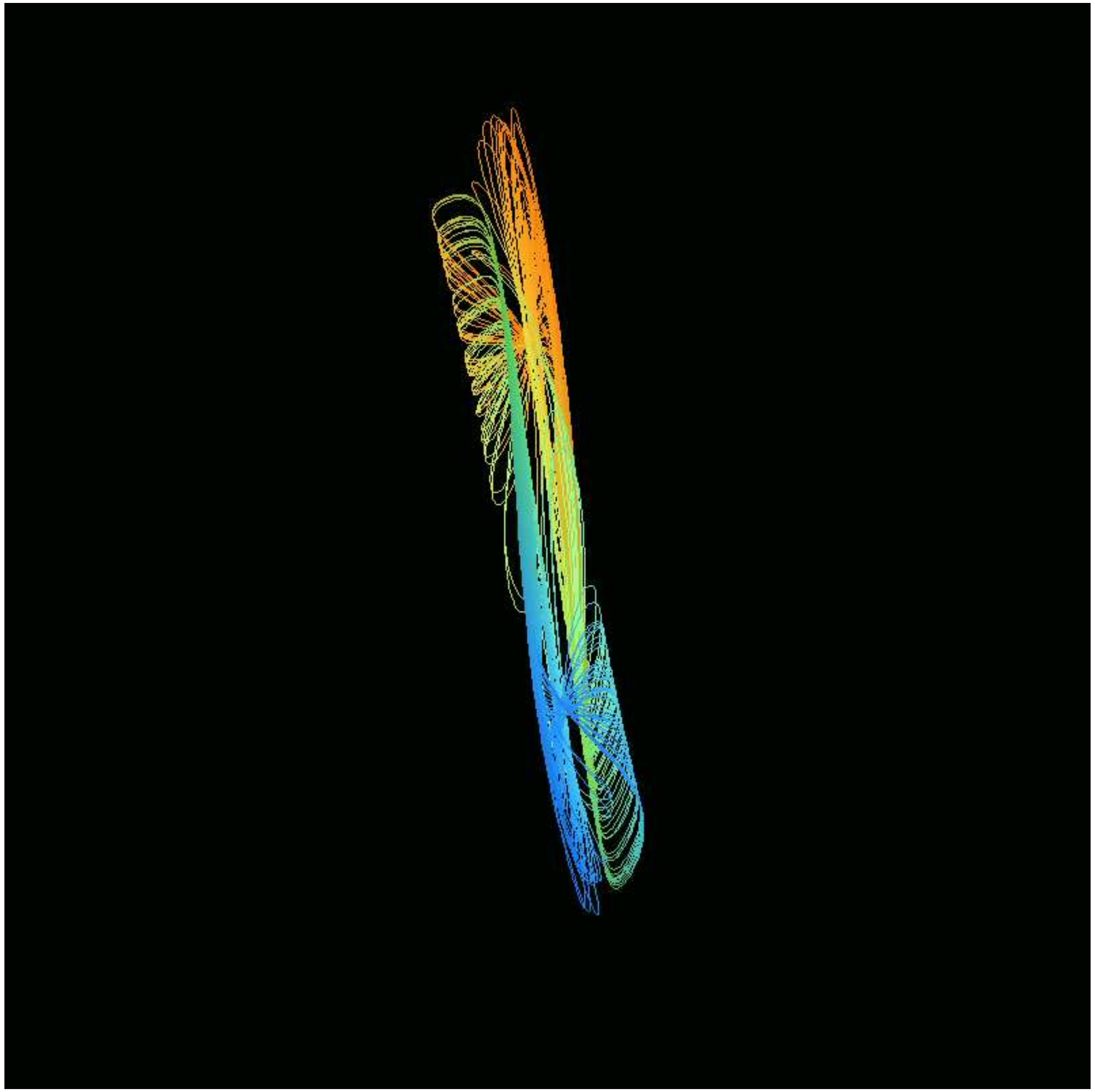




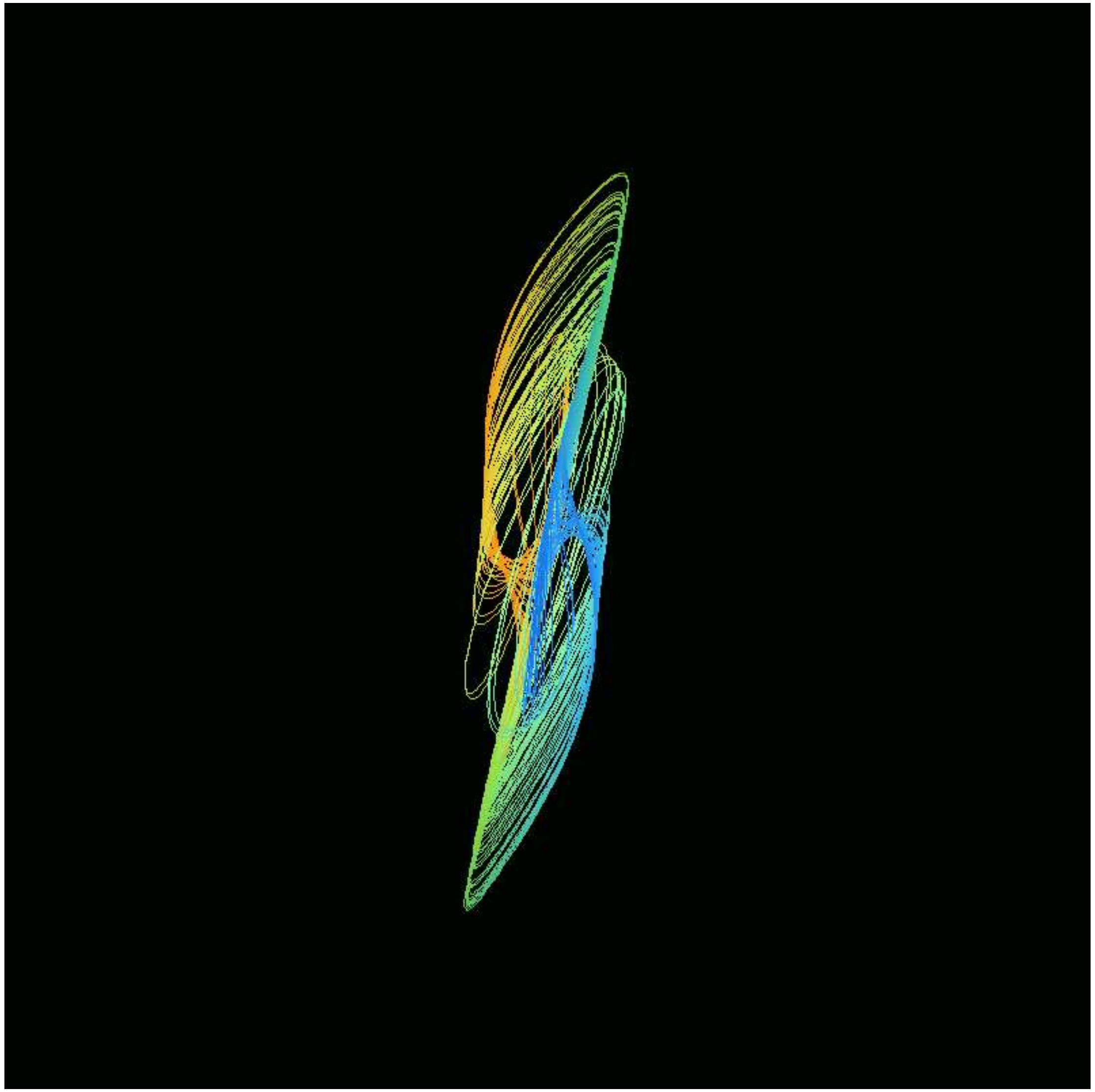
Projection into x_1-x_2-y_1—(1)



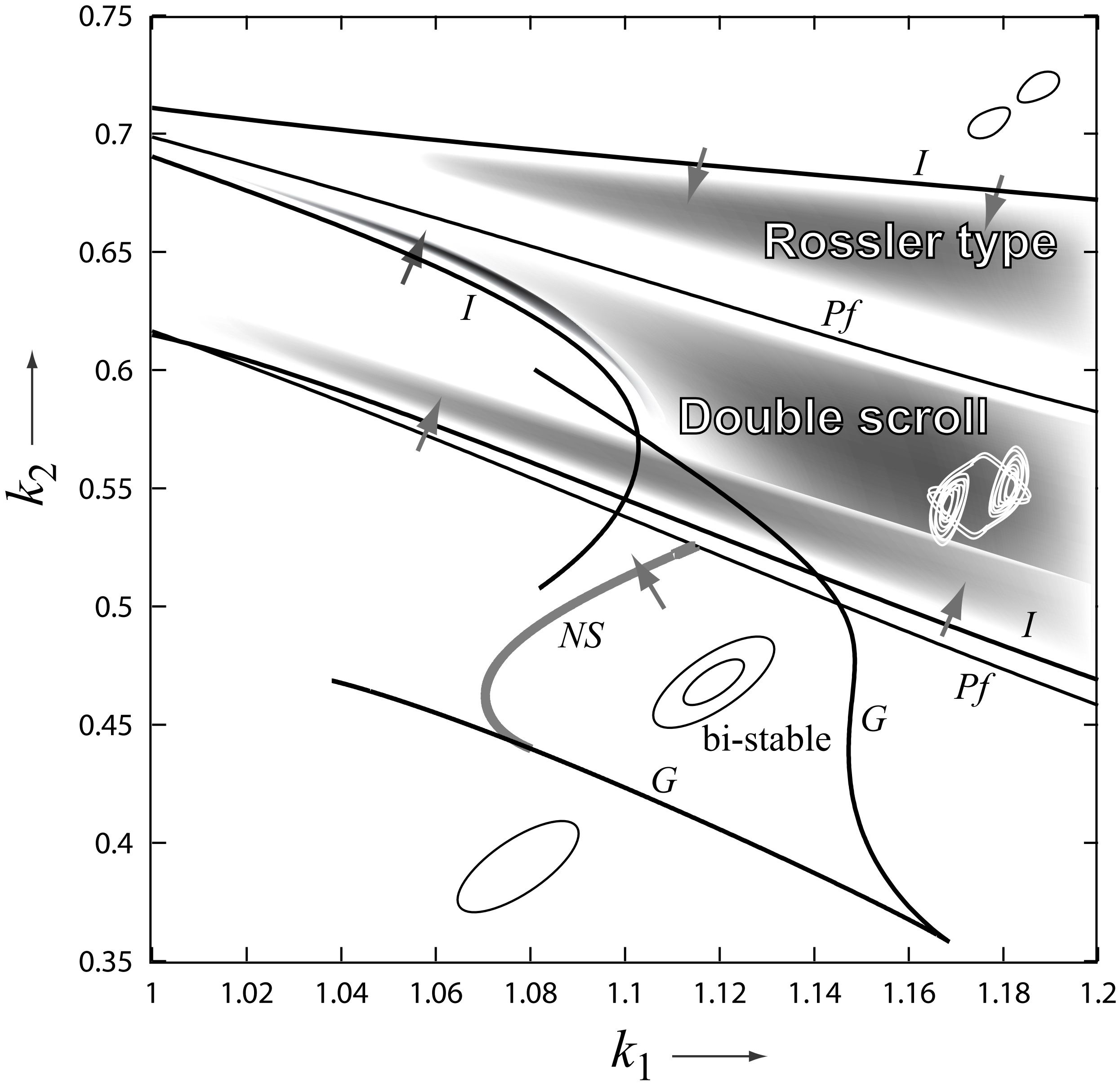
Projection into x_1-x_2-y_1—(2)

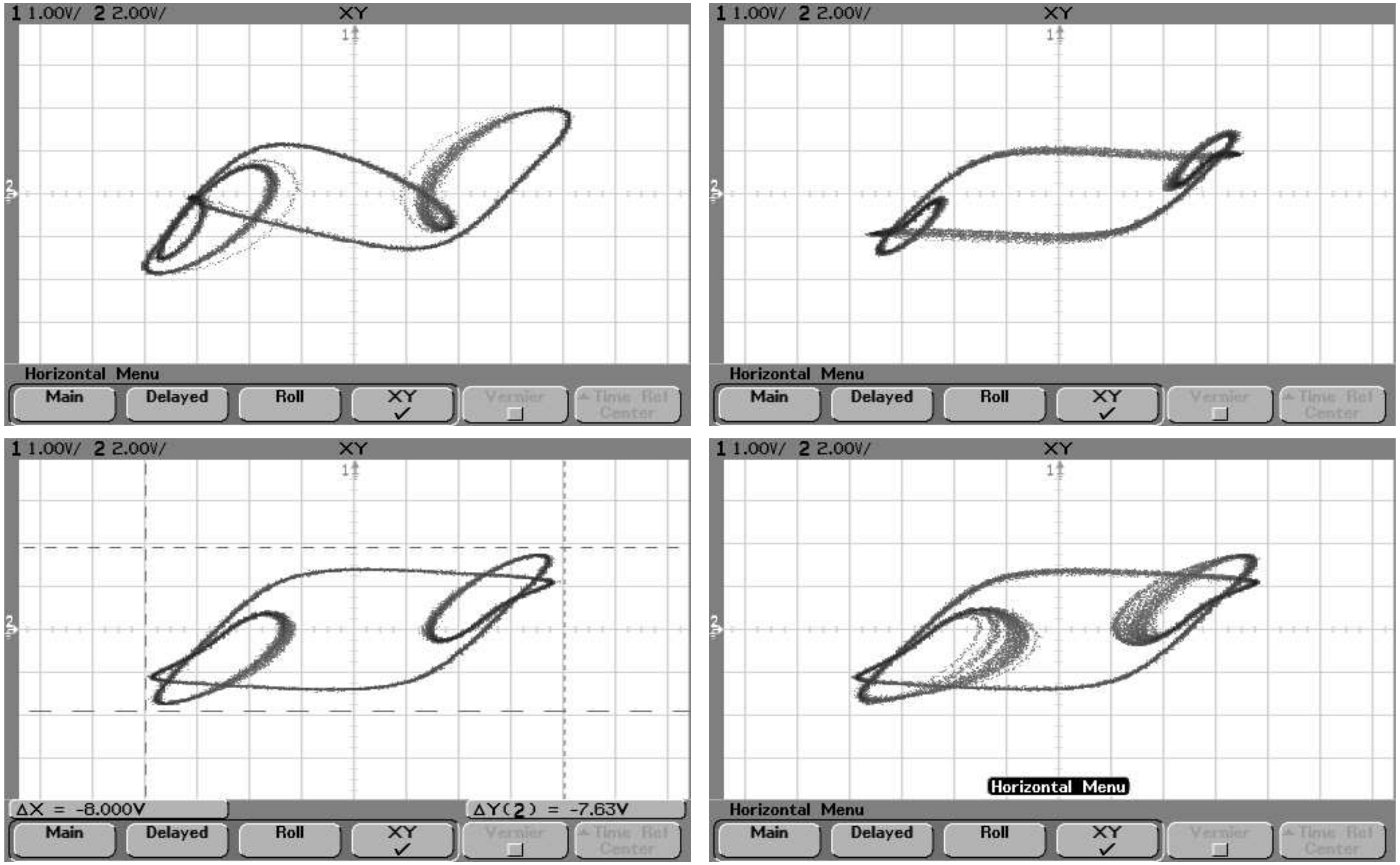


Projection into x_1-x_2-y_1—(3)

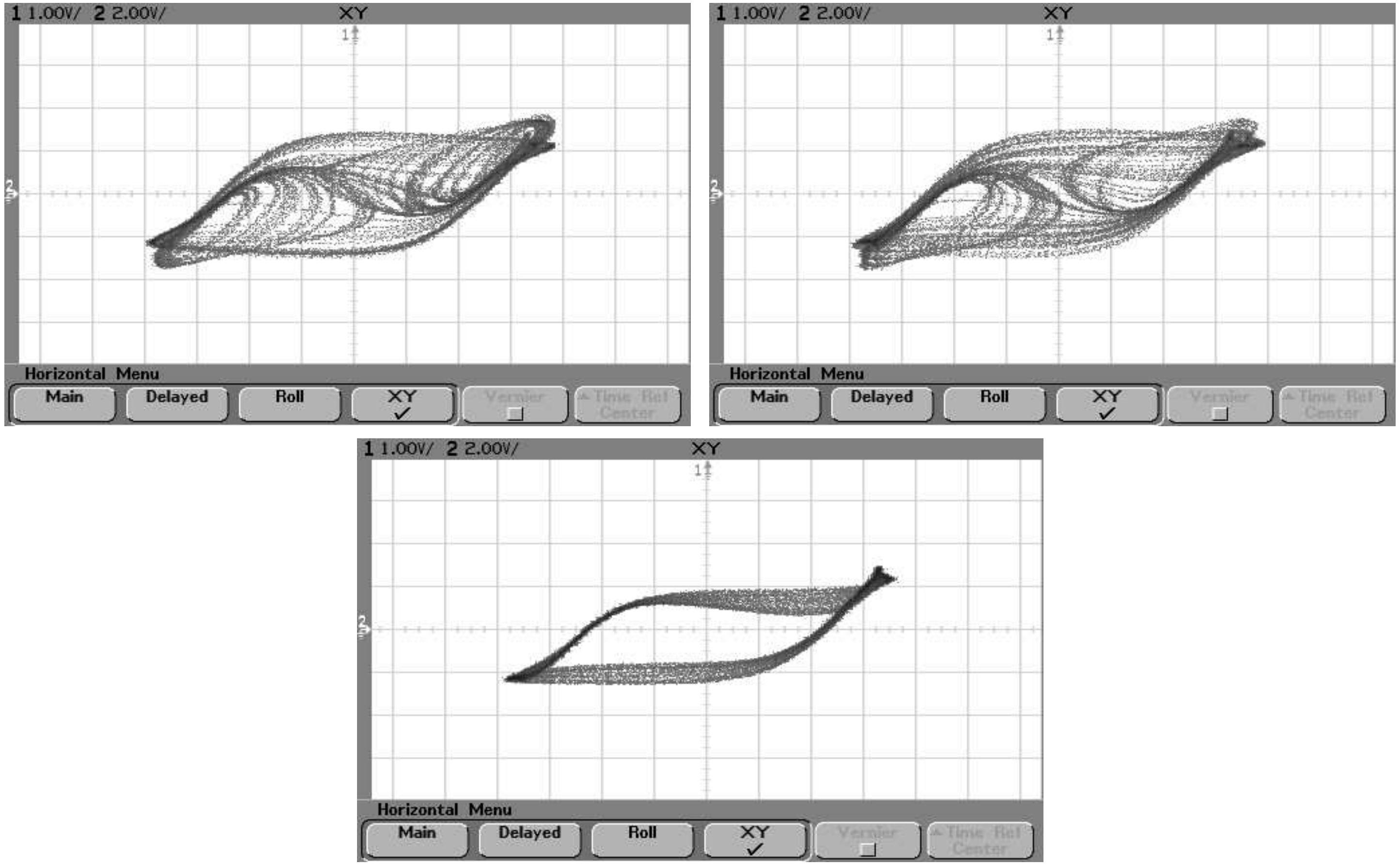


Enlargement of bifurcation diagram



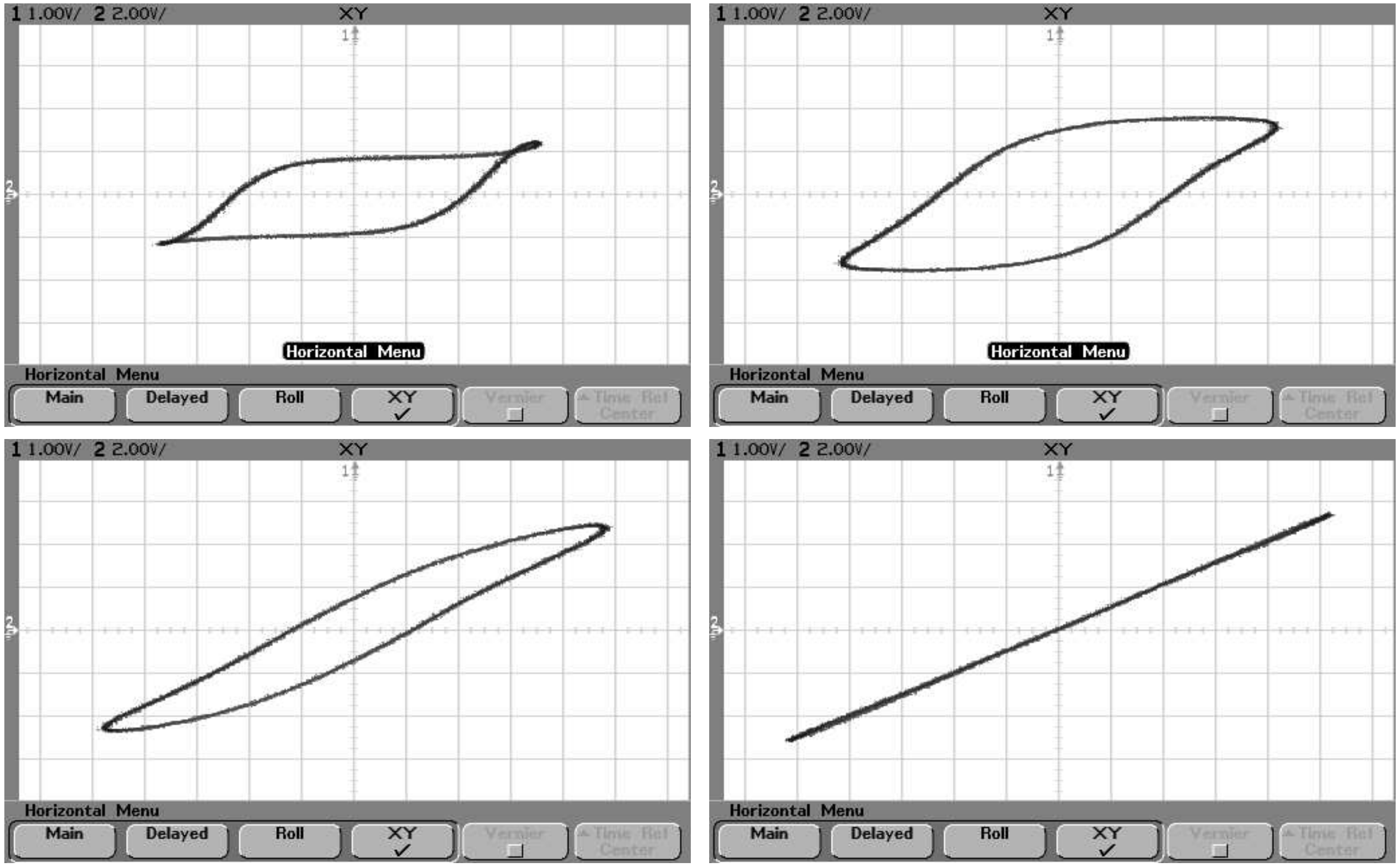


$$r_2 = 390 \to 360[\Omega]$$



The end of scrolling. left: $r_2 = 360[\Omega]$, right: $r_2 = 358[\Omega]$, lower: $r_2 = 346[\Omega]$.

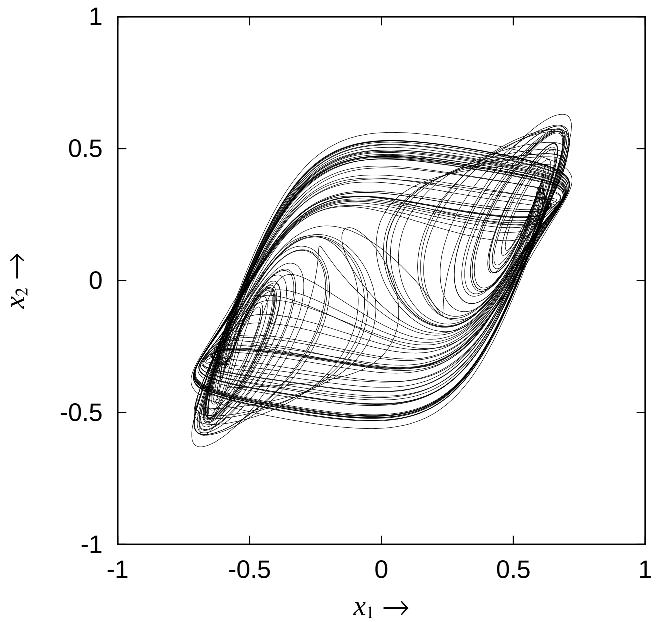
to be synchronized



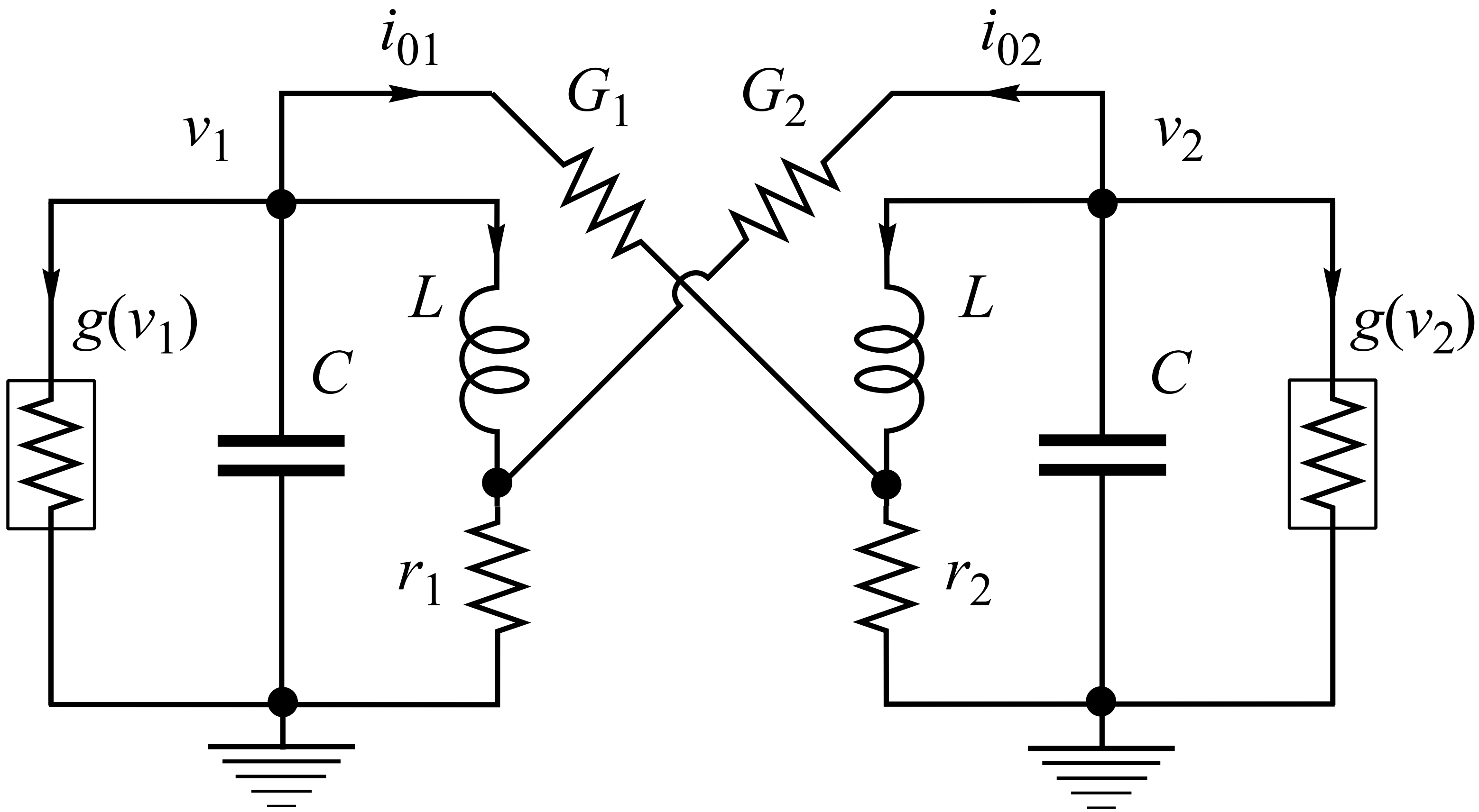
upper left: $r_2 = 342[\Omega]$, upper right: $r_2 = 340[\Omega]$.
lower left: $r_2 = 290[\Omega]$, lower right: $r_2 = 184[\Omega]$.

Features

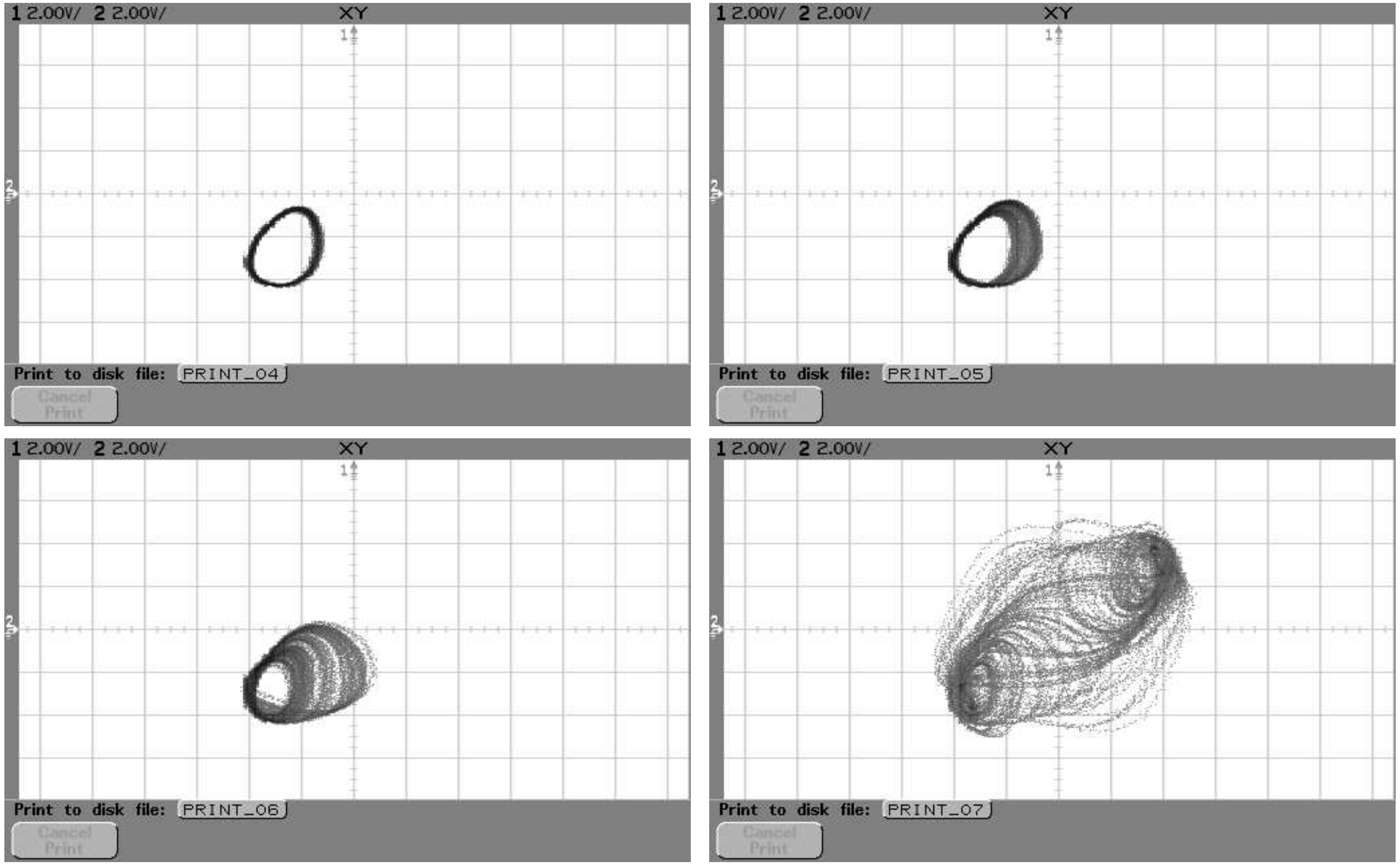
- There is no period-doubling route for $k1 = k_2$.
- (Osc. 1) equilibrium + (Osc. 2) limit cycle = (Osc. 1+2) chaotic solution.
- Cubic characteristics is essential: $g(v) = ax + bx^3$ with $a = -2.27 \times 10^{-3}, \quad b = 4.72 \times 10^{-5}$.



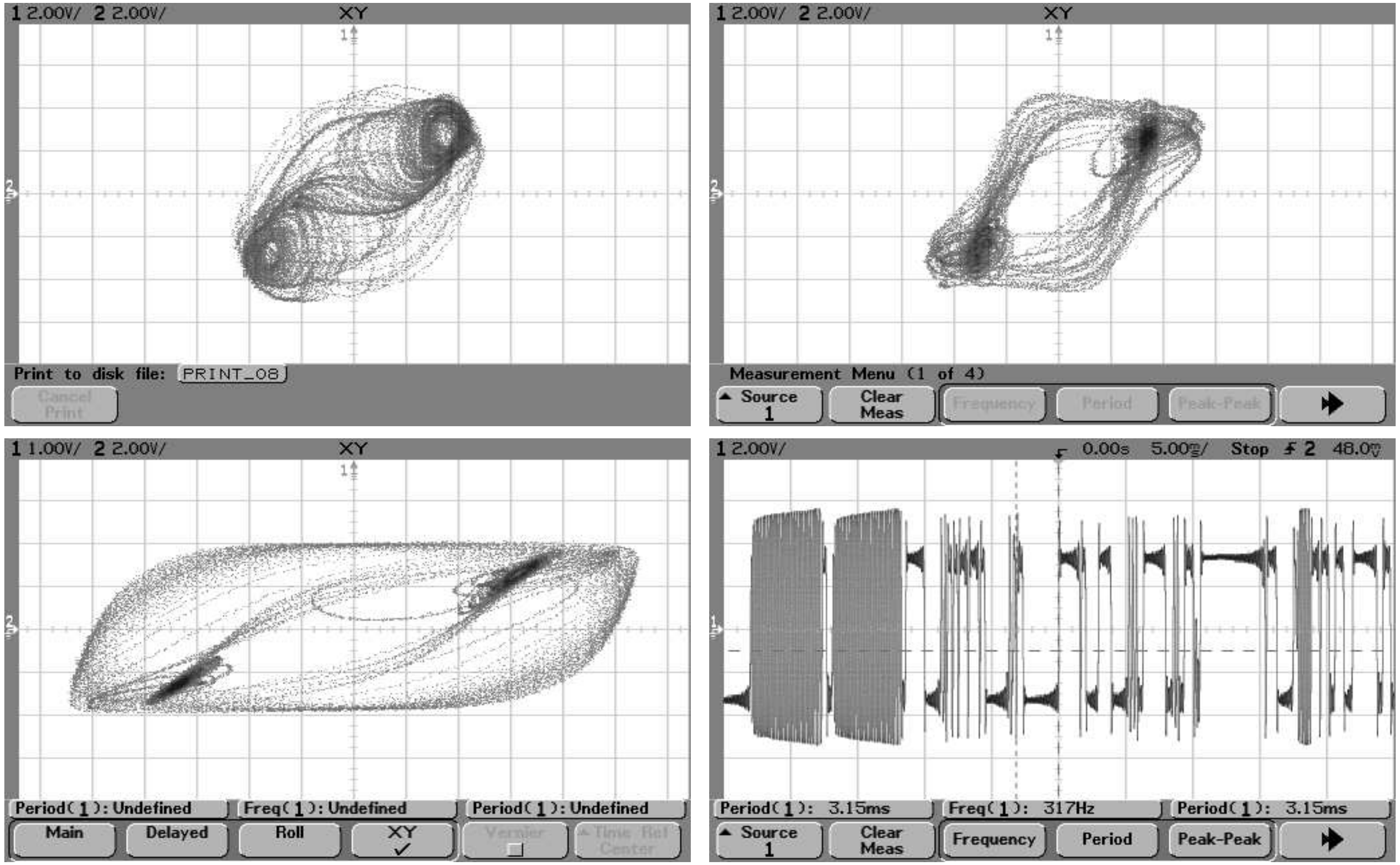
An application: Hybrid coupling



Period-doubling cascade



Chaotic response



Remarks

A resistively coupled BVP oscillators

- Bifurcation of periodic solutions
- Chaos is observed with $k_1 \neq k_2$.
- **No bifurcation for $k_1 = k_2$**

Future problems:

- synchronization of oscillators
- higher dimensional cases