An Aspect
of Oscillatory Conditions in Linear Systems
and Hopf Bifurcations in Nonlinear Systems

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Background

Oscillators — limit cycles in nonlinear circuits

- generation: lost stability for equilibrium — Hopf bifurcation
- required structural stability
- design problem
  - realization: RC phase shift or LC loop
  - design parameters: physical elements, wave shape and frequency
Oscillator in text books

1. provide an amplifier $A(j\omega)$ and a positive feedback $\beta(j\omega)$.
2. the trigger can be *noise* in the circuitry.
3. the output is applied into the input *in phase*.
4. no external force is needed anymore.
Transfer function expression

The transfer function of the feedback system:

\[ H(j\omega) = \frac{A(j\omega)}{1 - A(j\omega)\beta(j\omega)} \]

If \( A\beta \approx 1 \), \( \exists \omega = \omega_0 \), then \( H(j\omega_0) = \infty \).

Barkhausen criterion

\[ \text{Re}A(j\omega_0)\beta(j\omega_0) = 1, \quad \text{Im}A(j\omega_0)\beta(j\omega_0) = 0 \]
Polar plots

Barkhausen criterion of Nyquist plot:

\[\omega = \omega_0\]

\[\omega = \infty\]

\[\omega = 0\]

\[\text{Re} A(j\omega_0)\beta(j\omega_0) = 1, \quad \text{Im} A(j\omega_0)\beta(j\omega_0) = 0\]
Linear and nonlinear systems

Barkhausen criterion for linear systems gives a harmonic oscillation — **structurally unstable**!

\[ \downarrow \]

Need a gimmick to suppress amplitude growing.

- To retain the stability in large: **nonlinearity**— a natural assumption: limiter, clamper, thermistor

- getting the structural stability
Wien bridge oscillator

\[ R_4 \text{ should be nonlinear to keep stable oscillation.} \]
Wien bridge oscillator

\[ Z_1 Z_3 = Z_2 Z_4 \]
In this talk . . .

Design an oscillator — derive a condition for a given linear circuit.

We show that the following factors are equivalent:

- creation of zero impedance
- a current flows without power source — virtual source method
- Barkhausen criterion
- Hopf bifurcation
Virtual source method

Assumption of a virtual source by using the phasor method:

- Adding a virtual voltage (current) source into the system
- Zero impedance (admittance) makes a non-zero current (voltage) without source
Virtual voltage source

- place a virtual voltage source \( E = E_0e^{j\omega t} \) in an appropriate location in the circuit
- compute the whole impedance \( Z \)
- \( ZI = E \) shows "a condition to keep the current non-zero is \( Z = 0 \)."
Virtual current source

- place a virtual current source $I = I_0 e^{j\omega t}$ in an appropriate location in the circuit
- compute the whole admittance $Y$
- $YV = I$ shows “a condition to keep the voltage non-zero is $Y = 0$.”

These virtual source methods are equivalent to Hopf bifurcation analysis and Barkhausen criterion.
Example—LCR circuit

adding a virtual current source

\[ Z_1 = (r_1 + j\omega L_1) \parallel \frac{1}{j\omega C} \parallel R \]

\[ = \frac{R(r_1 + j\omega L_1)}{r_1 + R - \omega^2 R L_1 C + j\omega (L_1 + r_1 RC)} \]
LCR circuit

substitute $0 + j0$ into the denominator of $Z_1$

$$R + r_1 - \omega^2 RL_1 C = 0$$
$$\omega(L_1 + Rr_1 C) = 0$$

since $L_1 > 0$, $C > 0$, we have

$$\omega = \sqrt{\frac{R + r_1}{RL_1 C}}$$

frequency

$$Rr_1 = -\frac{L_1}{C}$$

Hopf bifurcation set

$R < 0$ required.
Hopf bifurcation analysis

Assume $R^{-1}v = g(v)$

$$C \frac{dv}{dt} = -g(v) - i, \quad L_1 \frac{di}{dt} = v - ri$$

Jacobian matrix:

$$J = \begin{pmatrix}
-\frac{1}{CR} & -\frac{1}{C} \\
\frac{1}{L_1} & -\frac{r}{L_1}
\end{pmatrix}.$$

where $R^{-1}$ is a linear part of $dg(v)/dt$
\[
\chi(\mu) = \det(A - \mu I) = 0
\]

\[
\chi(\mu) = \mu^2 + \left(\frac{1}{CR} + rL_1\right)\mu + \frac{R + r}{CRL_1} = 0
\]

Substitute **Hopf bifurcation condition**: \( \mu = 0 + j\omega \)
into above equation.
From the real and imaginary part, we have:

\[
\omega = \sqrt{\frac{R + r_1}{RL_1C}}
\]

\[
Rr_1 = -\frac{L_1}{C}
\]
Barkhausen criterion for BVP
as a bridge
Voltage feedback ratio

for the positive feedback loop:

\[ \beta = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2(1 - \omega^2 L_1 C + j\omega C r)}{r + R_2 - \omega^2 L_1 C R_2 + j\omega L_1 + j\omega C r R_2} \]

Let \( R = R_3 = R_4 \), and from \( \text{Re} A\beta = 1, \text{Im} A\beta = 0 \), we have:

\[ \omega = \sqrt{\frac{R + r_1}{R L_1 C}}, \quad R r_1 = -\frac{L_1}{C} \]

⇒ showing the same result.
Cautions

- Controlling Hopf bifurcation by choosing parameter values
- No guarantee for global stability of the limit cycle
- post-process: getting global stability, e.g., replacing $R$ by a nonlinear conductor
- Frequency and wave shape — simulation is needed
Application — extension of BVP

virtual source method:

\[ Z_2 = Z_1 \parallel (r_2 + j\omega L_2) \]

\[ R(r_1 + r_2) + r_1 r_2 - \omega^2 (r_2 R L_1 C + L_2 (r_1 R C + L_1)) = 0 \]

\[ R(L_1 + L_2) + r_1 L_2 + r_2 L_1 + r_1 r_2 R C - \omega^2 R L_1 L_2 C = 0 \]
We have Hopf bifurcation curve \( H(r_1, r_2) \), and

\[
\omega = \sqrt{\frac{(r_1 + r_2)R + r_1 r_2}{L_1 L_2 + r_2 RCL_1 + r_1 RL_2 C}}
\]

\( r_2 \to \infty \)

\[
\omega_\infty = \sqrt{\frac{R + r_1}{R L_1 C}}
\]

\( r_2 \to 0 \)

\[
\omega_0 = \sqrt{\frac{R r_1}{L_1 L_2 + R r_1 L_2 C}}
\]
Slow oscillation (6 kHz)

\[ C = 0.022[\mu F], \quad r_1 = 500[\Omega], \quad L_1 = 10[mH], \quad L_2 = 1[mH] \]
oscillation dead

\[ C = 0.022[\mu F], \quad r_1 = 500[\Omega], \quad L_1 = 10[mH], \quad L_2 = 1[mH] \]

(a) 400 Ω
revival

\[ C = 0.022[\mu F], \quad r_1 = 500[\Omega], \quad L_1 = 10[mH], \quad L_2 = 1[mH] \]
Fast oscillation (34 kHz)

\[ C = 0.022[\mu F], \quad r_1 = 500[\Omega], \quad L_1 = 10[mH], \quad L_2 = 1[mH] \]

(a) 100 \, \Omega
Hopf bifurcation set in $k_1$-$k_2$ plane

- Slow-Oscillatory
- Quiescent
- Fast-Oscillatory
Bifurcation of Hopf bifurcation set

(a) $\alpha = 0.1$. 

(b) $\alpha = 0.14$. 

(c) $\alpha = 0.145$. 

(d) $\alpha = 0.15$. 
\[ \alpha = 0.18 \]
extended BVP circuit 1/5

\[ Z = R \parallel \frac{1}{j\omega C_1} \parallel \left( j\omega L + \left( \frac{1}{j\omega C_2} \parallel r \right) \right) \]
extended BVP circuit 2/5

\[ Z = \frac{R(r - \omega^2 C_2 r L + j\omega L)}{R(1 + j\omega C_2 r) + (1 + j\omega C_1 R)(r - \omega^2 C_2 r L + j\omega L)} \]

The denominator \( \equiv 0 + j0 \)

\[ R + r - \omega^2 (C_2 r - C_1 R)L = 0 \]

\[ (C_1 + C_2) r R + L - \omega^2 C_1 C_2 r R L = 0 \]
extended BVP circuit 3/5

Frequency:

\[ \omega = \sqrt{\frac{L + (C_1 + C_2)rR}{C_1C_2rRL}} \]

Bifurcation set:

\[(L + (C_1 + C_2)rR)(C_2r + C_1R)L - (R + r)C_1C_2rRL = 0\]

\(\Rightarrow\) gives Hopf bifurcation set in \(r-R\) plane.
extended BVP circuit 4/5

\[ r = \frac{-C_2L - C_1^2R^2 \pm \sqrt{-4C_1C_2^2LR^2 + (C_2L + C_1^2R^2)^2}}{2C^2R} \]

In case that \( C_1 = C_2 = C \):

\[ rR = -\frac{L}{C}, \quad r = -R \]
For more detailed analysis, numerical computation is needed.
REMARKS

Oscillator design:

- Hopf bifurcation analysis, Barkhausen criterion, virtual source method
- Hopf bifurcation set: oscillation condition curve itself
- Phasor method is easier to analyze the system