Calculation of the Isocline for the Fixed Point with a Specified Argument of Complex Multipliers

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Background

Analysis of nonlinear dynamical system described by ODE or DE

Equilibria or fixed/periodic points:

- location
- stability $\longrightarrow$ bifurcation

Calculation of periodic points: solving boundary value problem

- shooting method; Newton's method
- collocation method
- automatic differentiation

Proposal

1. Calculation of isocline: a fixed argument of the complex multipliers
   - simple shooting algorithm
   - classify topological property of periodic points
2. an application of calculation: **calculation of Neimark-Sacker bifurcation**.

Description of the Problem

A discrete map T:

$$T : \boldsymbol{R}^n \to \boldsymbol{R}^n; \quad \boldsymbol{u} \mapsto T(\boldsymbol{u}), \quad \boldsymbol{u} \in \boldsymbol{R}^n$$

- Parameters: $\lambda_1 \in \boldsymbol{R}$, $\lambda_2 \in \boldsymbol{R}$
- T is C^∞ for $\boldsymbol{u}$, λ_1, and λ_2.

The fixed point $\boldsymbol{u}_0$:

$$T(\boldsymbol{u}_0) = \boldsymbol{u}_0.$$

Complex multipliers

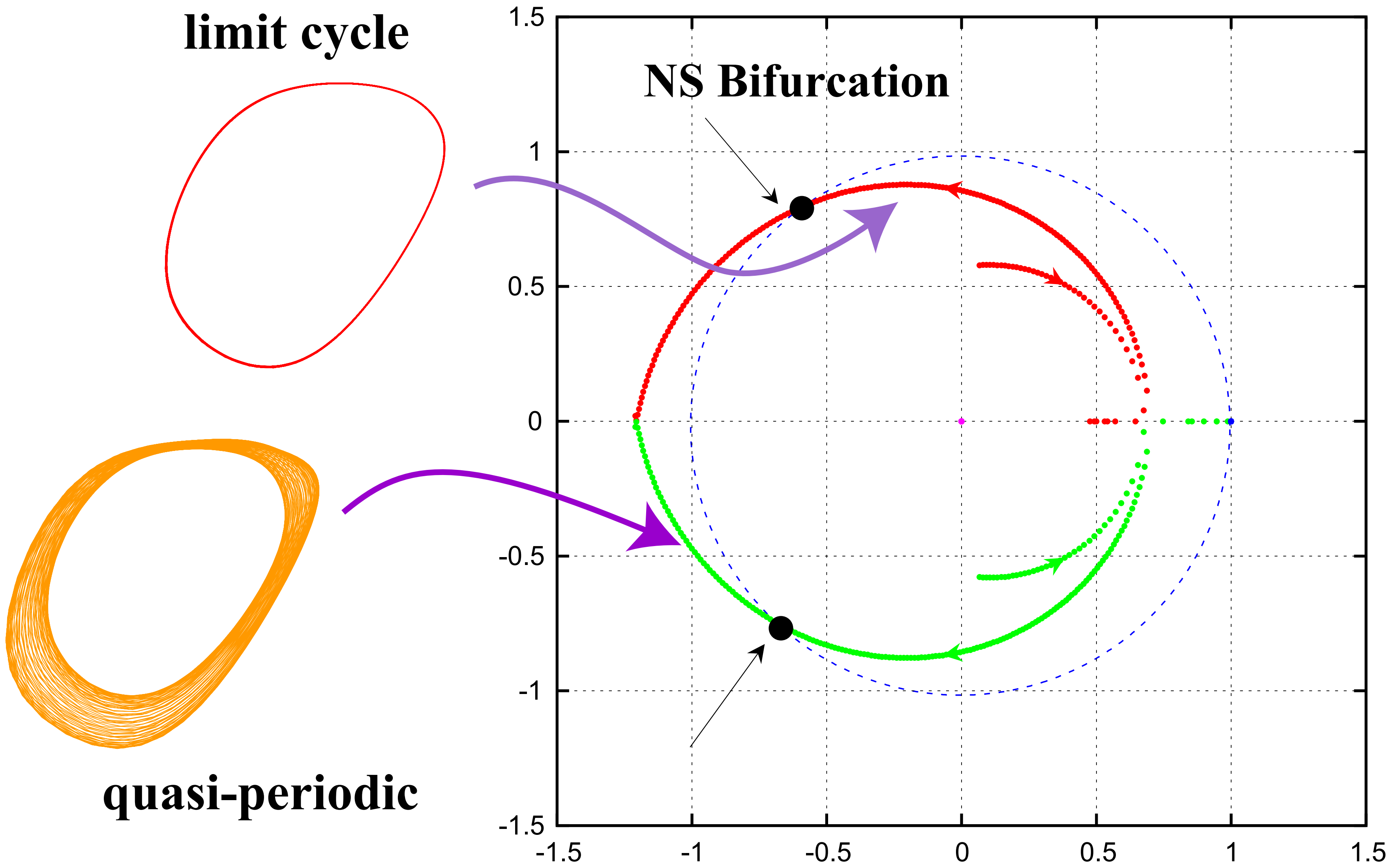
A pair of complex conjugate multipliers:

$$\mu, \bar{\mu} = re^{\pm i\theta} = r(\cos\theta \pm i\sin\theta).$$

r: the radius, θ: the argument.

r and θ determine stability and angular velocity of the local linear space about u_0.

Complex multipliers



Calculation of the fixed point

$\boldsymbol{J}$: the Jacobian matrix of T at $\boldsymbol{u}_0$
The characteristic equation:

$$\chi = \det[\boldsymbol{J} - re^{i\theta}] = \Re\chi + i\Im\chi = 0$$

Condition of the fixed point with θ

$$\boldsymbol{F} = \begin{pmatrix} T(\boldsymbol{u}) - \boldsymbol{u} \\ \Re\chi \\ \Im\chi \end{pmatrix} = \boldsymbol{0}$$

Variables: $(\boldsymbol{u}, \lambda_1, r) \in \boldsymbol{R}^{n+2}$.

Jacobian matrix

Then the Jacobian matrix for Newton's method:

$$DF = \begin{pmatrix} \dfrac{\partial T}{\partial \boldsymbol{u}_0} - I_n & \dfrac{\partial T}{\partial \lambda_1} & \dfrac{\partial T}{\partial r} \\ \dfrac{\partial \Re\chi}{\partial \boldsymbol{u}_0} & \dfrac{\partial \Re\chi}{\partial \lambda_1} & \dfrac{\partial \Re\chi}{\partial r} \\ \dfrac{\partial \Im\chi}{\partial \boldsymbol{u}_0} & \dfrac{\partial \Im\chi}{\partial \lambda_1} & \dfrac{\partial \Im\chi}{\partial r} \end{pmatrix}$$

The variational equation of T

For ODE:

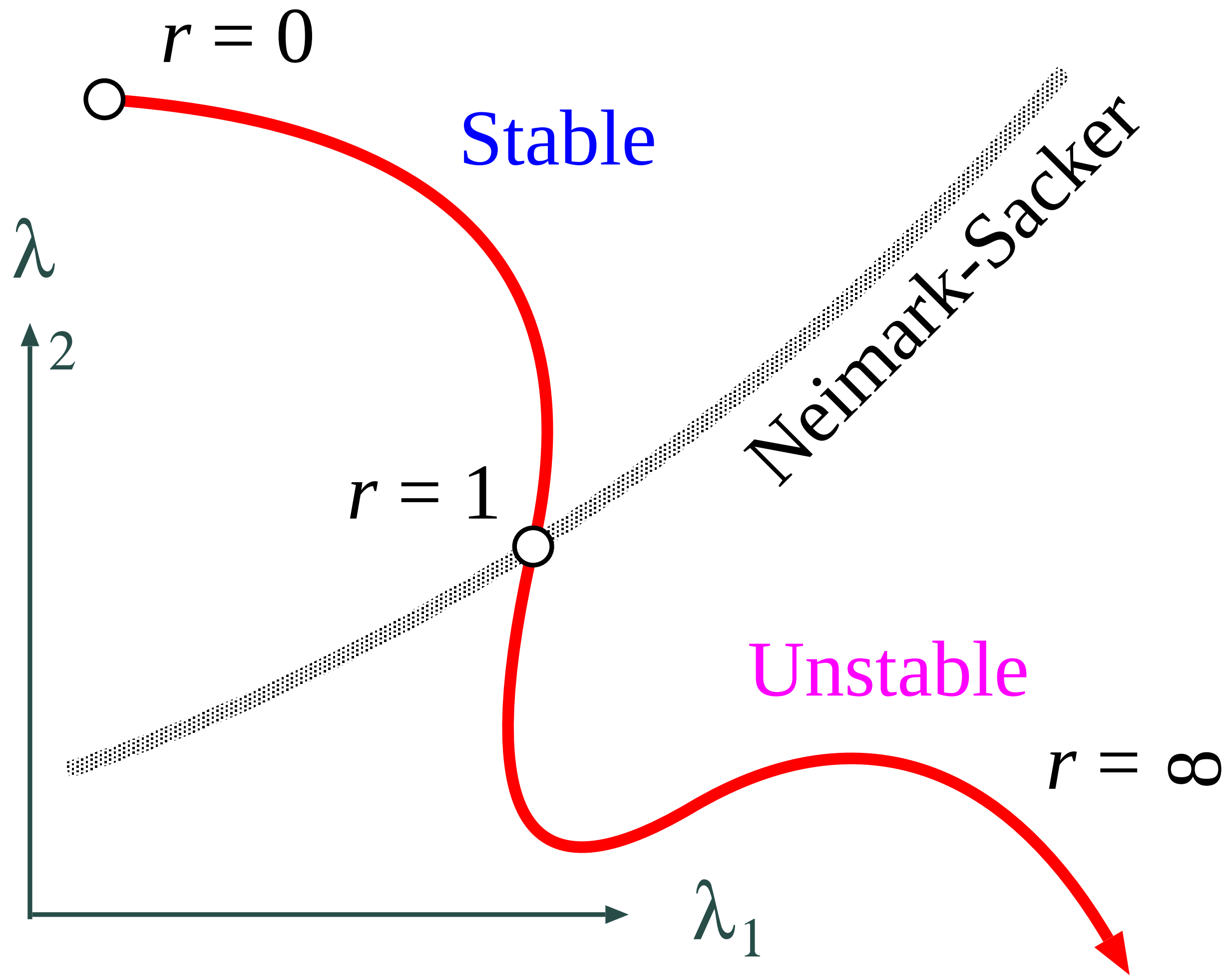
$$\frac{d}{dt}\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0}$$

$$\frac{d}{dt}\frac{\partial \boldsymbol{\varphi}}{\partial \lambda_1} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} + \frac{\partial \boldsymbol{f}}{\partial \lambda_1}$$

can be obtained by numerical numerical integration like Runge-Kutta method.

Isocline

For the fixed $\theta \in [0, \pi]$, $(\boldsymbol{u}_0, \lambda_1, r)$ forms an isocline in the parameter plane λ_1-λ_2.



Properties

- $\theta = 0$ or $\theta = \pi$: critical line between real and complex multipliers. (nodal or rotational linear space about $\boldsymbol{u}_0$.)
- any rational ratio with π of θ: deeply related to the NS bifurcation and frequency entrainment regions.

Calculation of NS bifurcation

Exchange variables and parameters: $r \rightleftarrows \theta$
Let $r = 1$, then solve

$$\boldsymbol{F} = \begin{pmatrix} T(\boldsymbol{u}) - \boldsymbol{u} \\ \Re\chi \\ \Im\chi \end{pmatrix} = \mathbf{0}$$

for $(\boldsymbol{u}, \lambda_1, \lambda_2) \in \boldsymbol{R}^{n+2}$

The Jacobian matrix

$$DF = \begin{pmatrix} \dfrac{\partial T}{\partial \boldsymbol{u}_0} - I & \dfrac{\partial T}{\partial \lambda_1} & \dfrac{\partial T}{\partial \lambda_2} \\ \dfrac{\partial \Re\chi}{\partial \boldsymbol{u}_0} & \dfrac{\partial \Re\chi}{\partial \lambda_1} & \dfrac{\partial \Re\chi}{\partial \lambda_2} \\ \dfrac{\partial \Im\chi}{\partial \boldsymbol{u}_0} & \dfrac{\partial \Im\chi}{\partial \lambda_1} & \dfrac{\partial \Im\chi}{\partial \lambda_2} \end{pmatrix}$$

Features

Neimark-Sacker bifurcation — explicitly parameterized by θ.

- Very simple formulation to calculate NS bifurcation.
- Codimension-two bifurcation parameter values can be obtained, $\theta = 0$ and $\theta = \pi$.
- No predictor-corrector method is needed.
- Not sensitive for stability of the periodic point

Example 1

A Discrete Chaotic Map

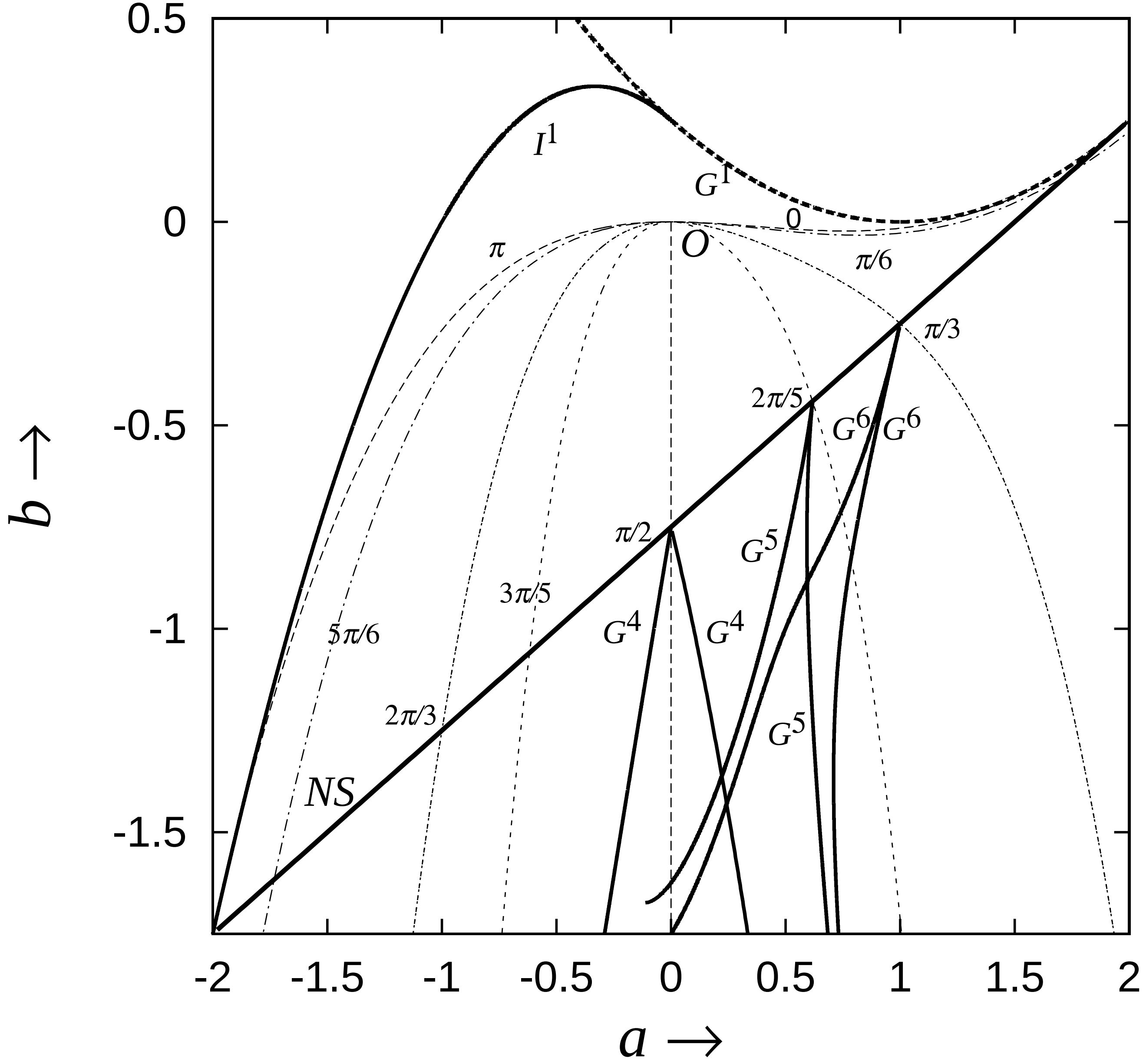
$$\begin{cases} x_1(k+1) = x_2(k) + a x_1(k) \\ x_2(k+1) = x_1^2(k) + b \end{cases} \qquad k = 0, 1, 2, \cdots$$

The characteristic equation :

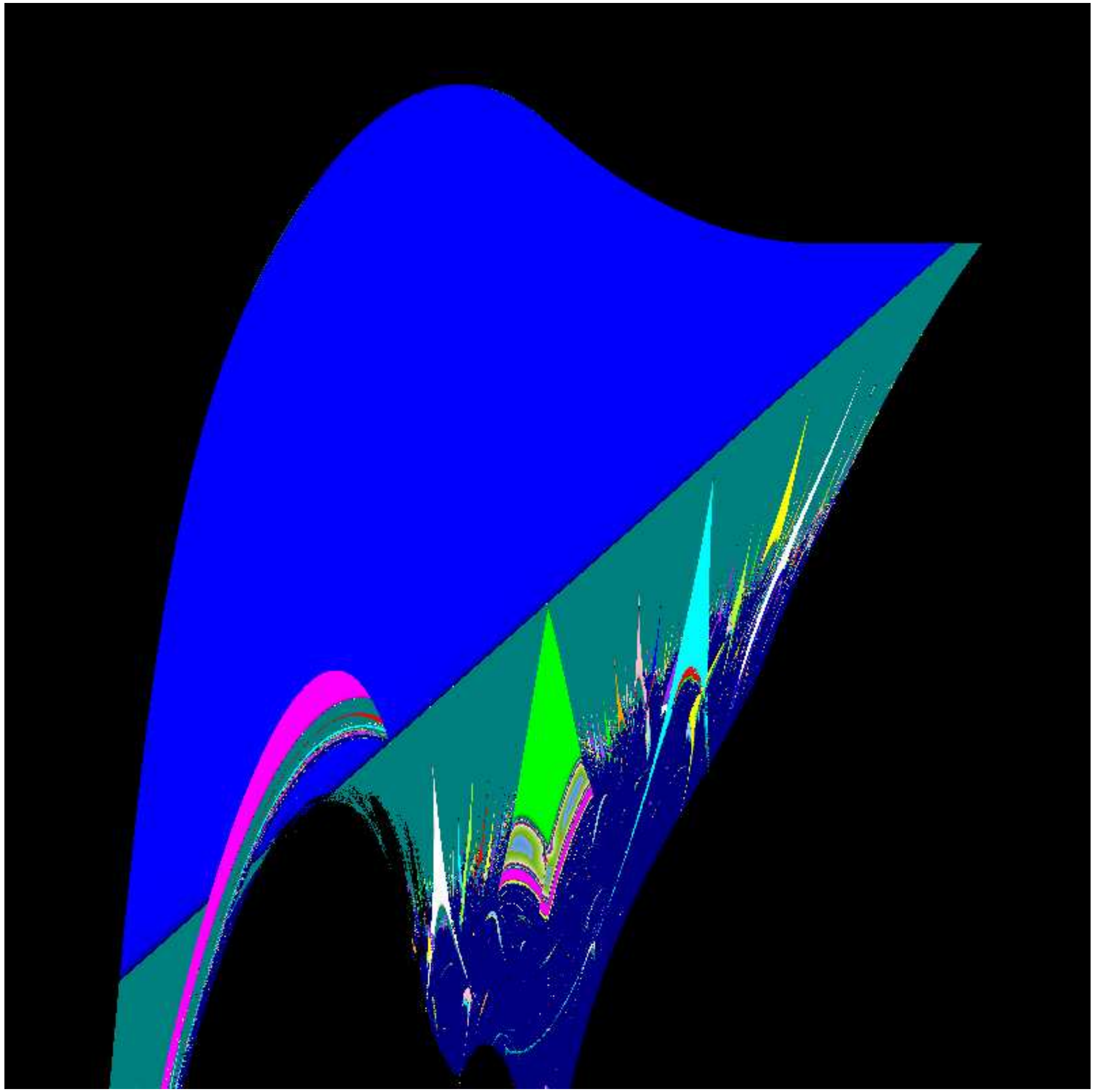
$$\Re\chi = r^2(\cos^2\theta - \sin^2\theta) - r\cos\theta\,\mathrm{tr}\,\boldsymbol{X} + \det\boldsymbol{X}$$
$$\Im\chi = 2r\cos\theta - \mathrm{tr}\,\boldsymbol{X}$$

$\boldsymbol{X}$: the fundamental solution matrix of the variational equation.

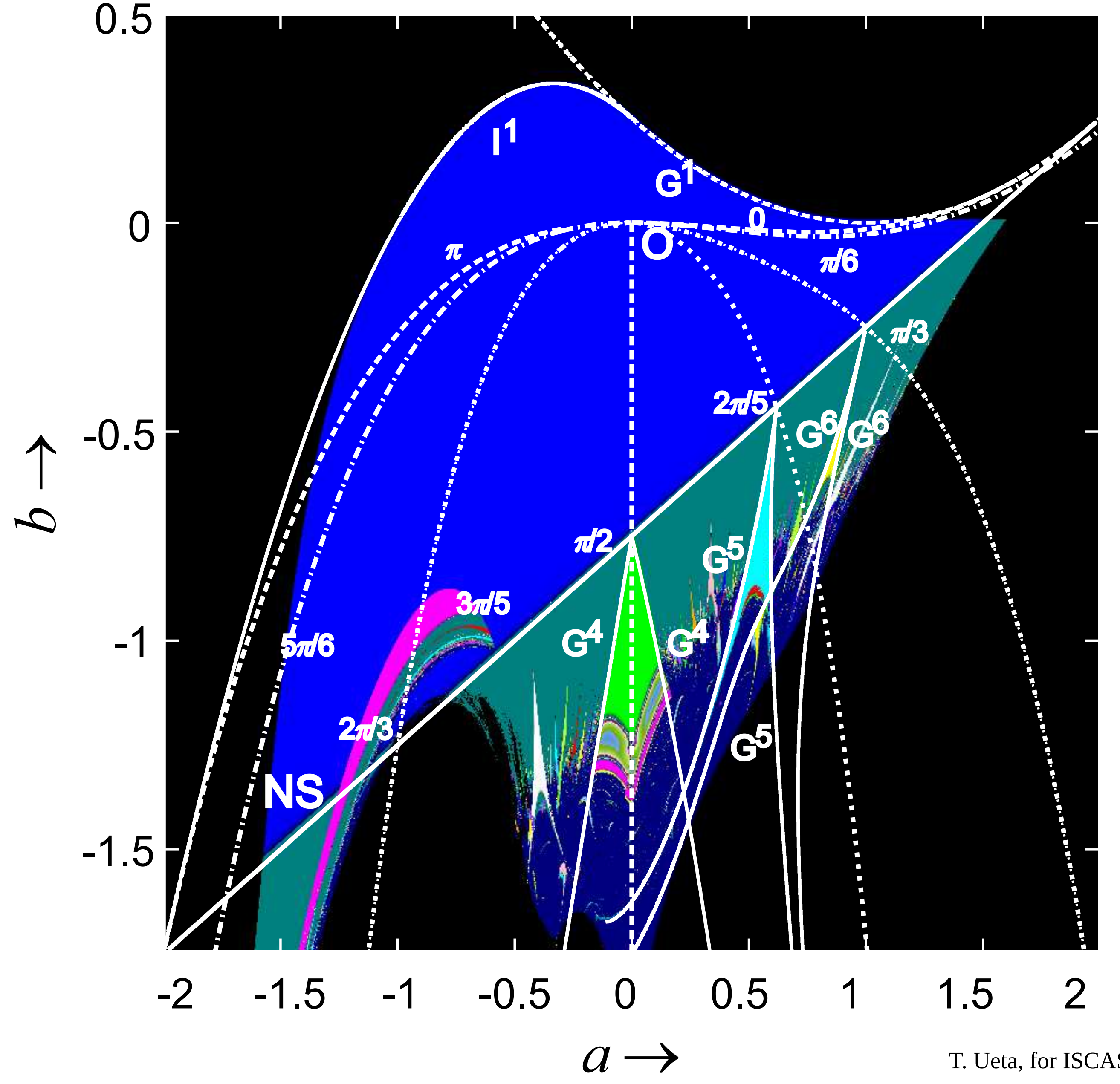
Bifurcation diagram



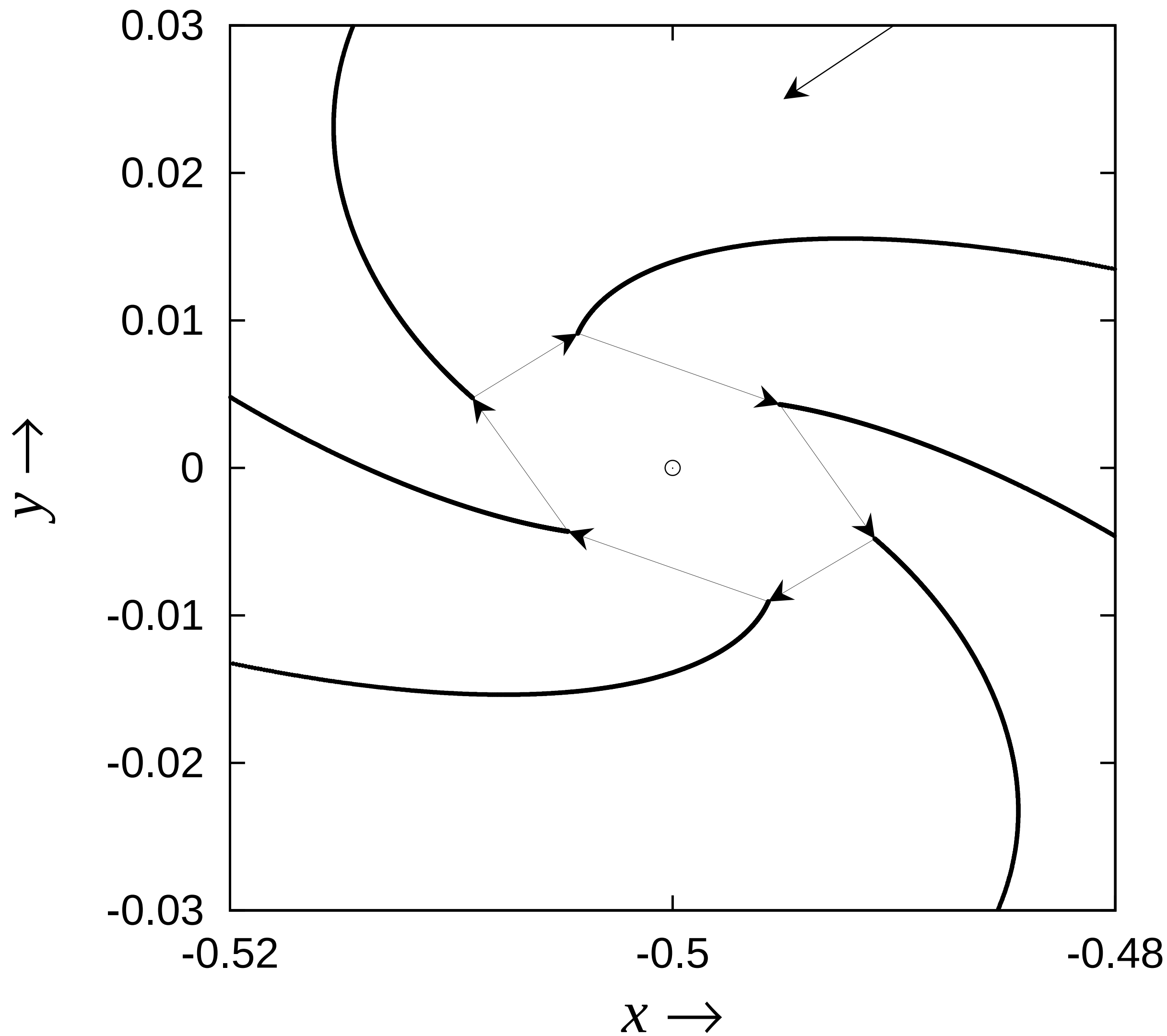
Bifurcation diagram



NS bifurcation: $b = 0.5a - 0.75$

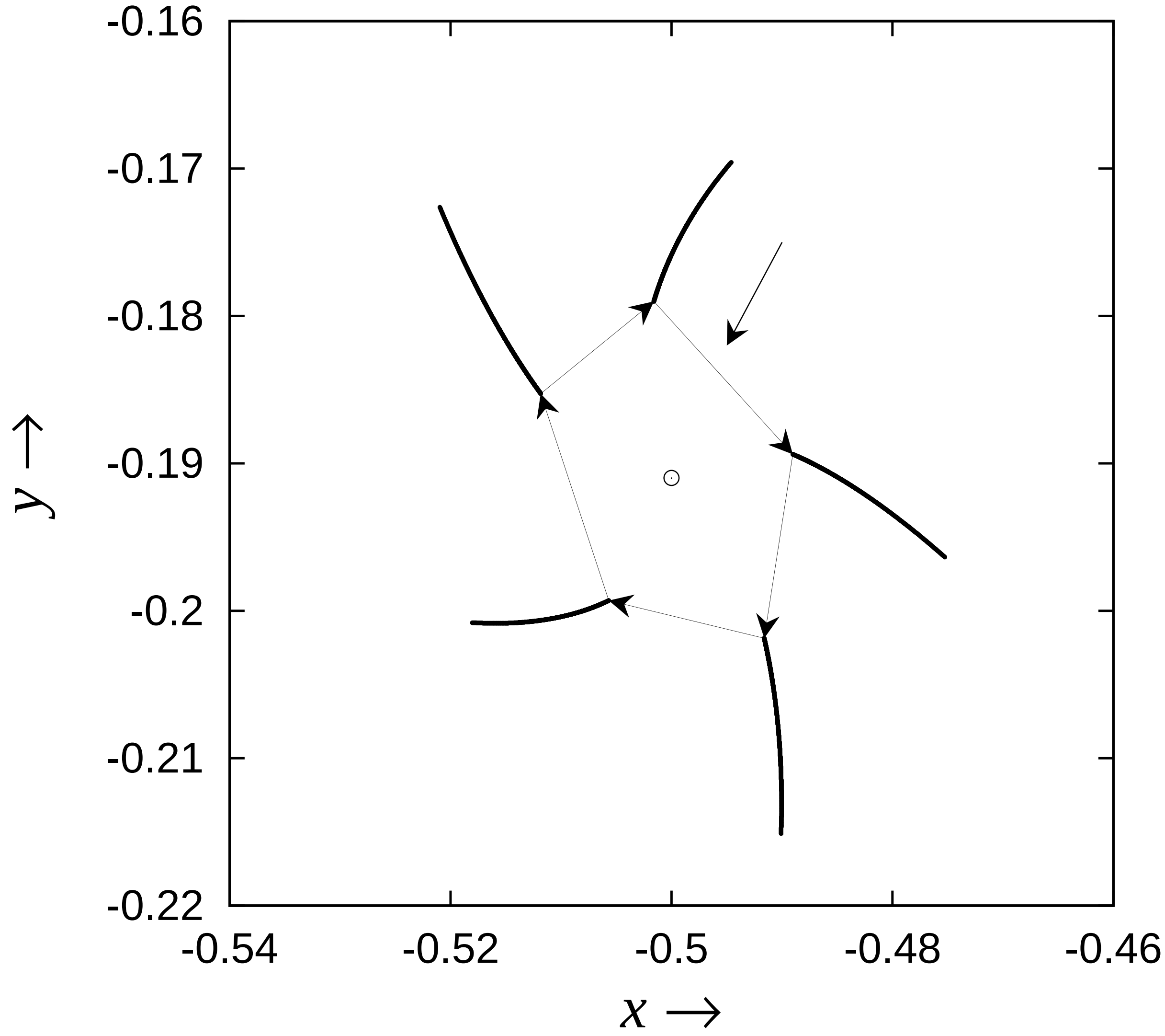


Period-6, $\theta = \pi/3$



$$a = 1.0,\, b = -0.25,\, \boldsymbol{u}_0 = (-0.5, 0.0).$$

period-5, $\theta = 2\pi/5$



$$a = 0.618033, \, b = -0.440983,$$

A couple of neurons

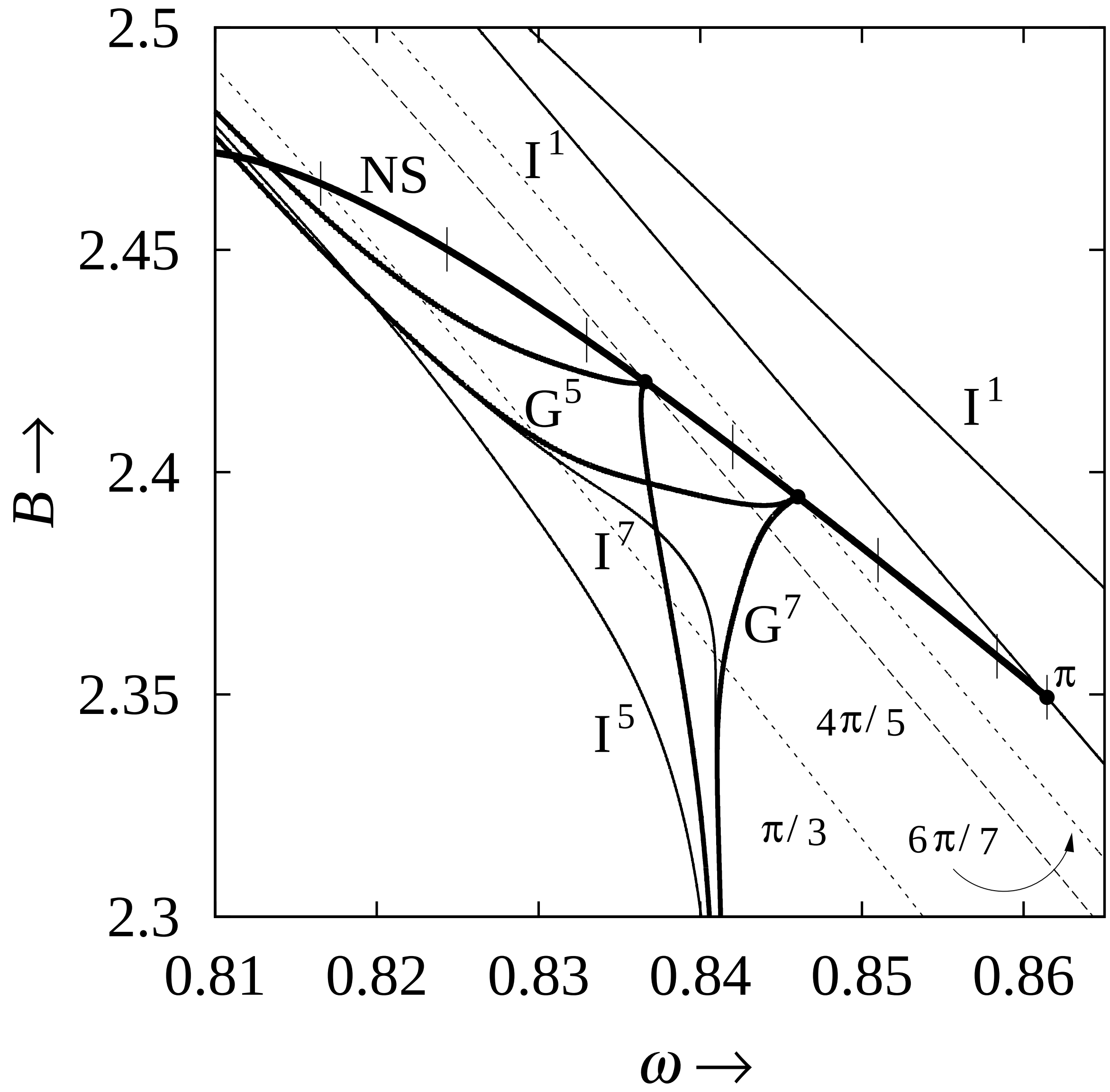
A system described by ODE:

$$\begin{cases} \dot{x}_1 &= 10f(x_1) - 10f(x_2) - 2.5 + B\cos\omega t \\ \dot{x}_2 &= 10f(x_1) + 2f(x_2) - 9.0 \end{cases}$$

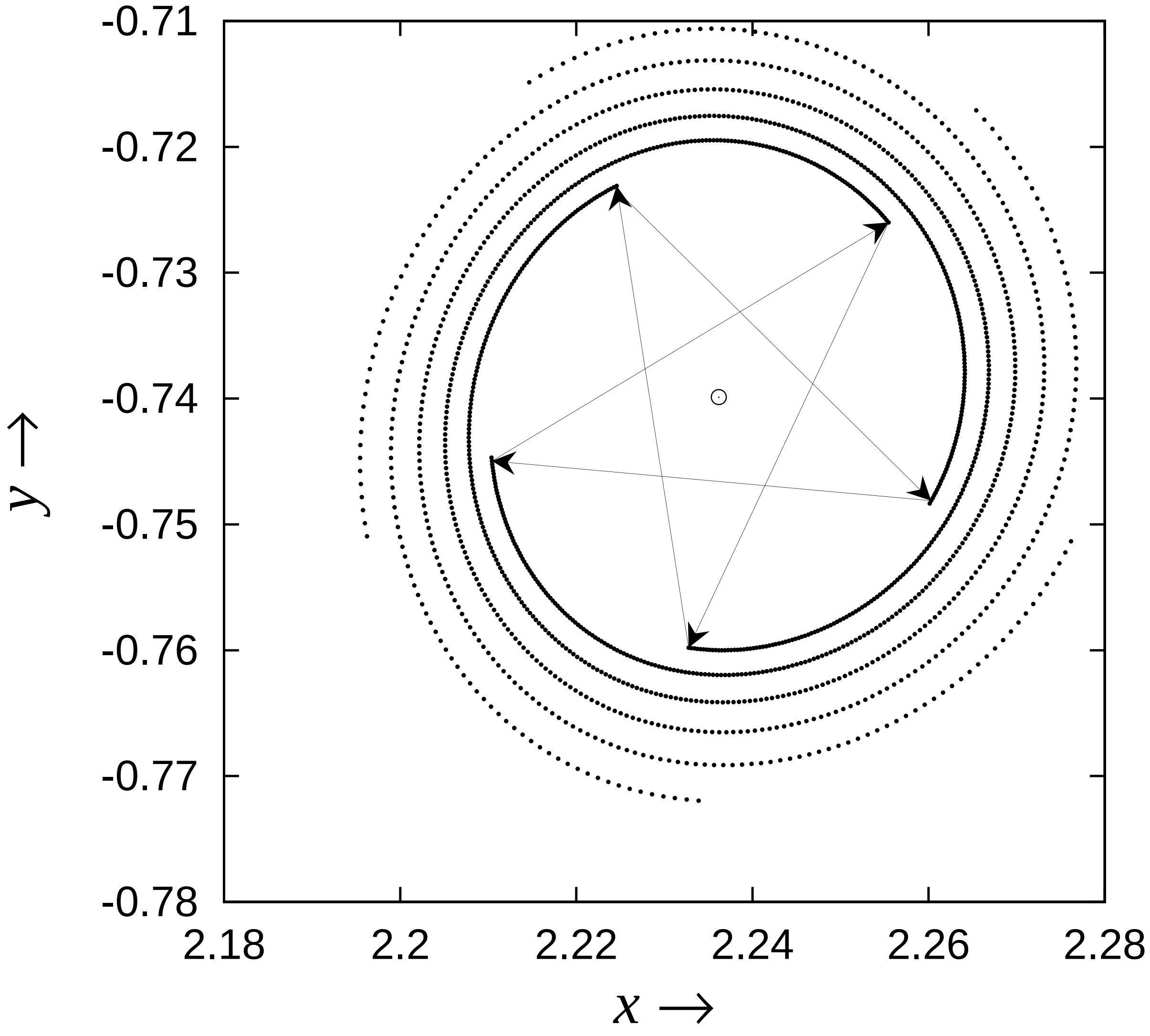
$$f(x) = \frac{1}{1+e^{-x}}$$

- quasi-periodic solutions
- chaotic motion
- frequency entrainment

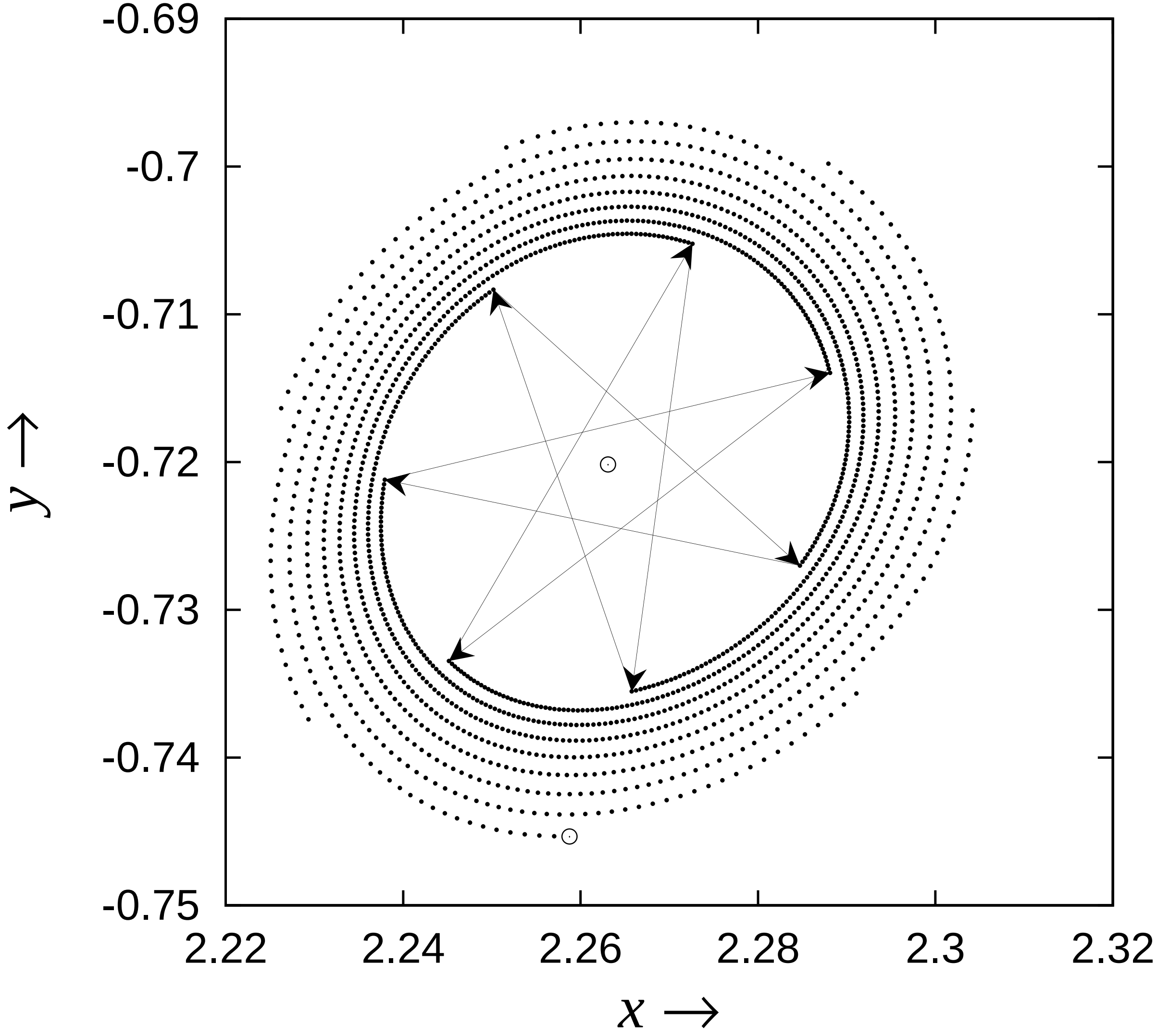
Bifurcation diagram



Period-5, $\theta = 2\pi/5$



Period-6, $\theta = 2\pi/7$.



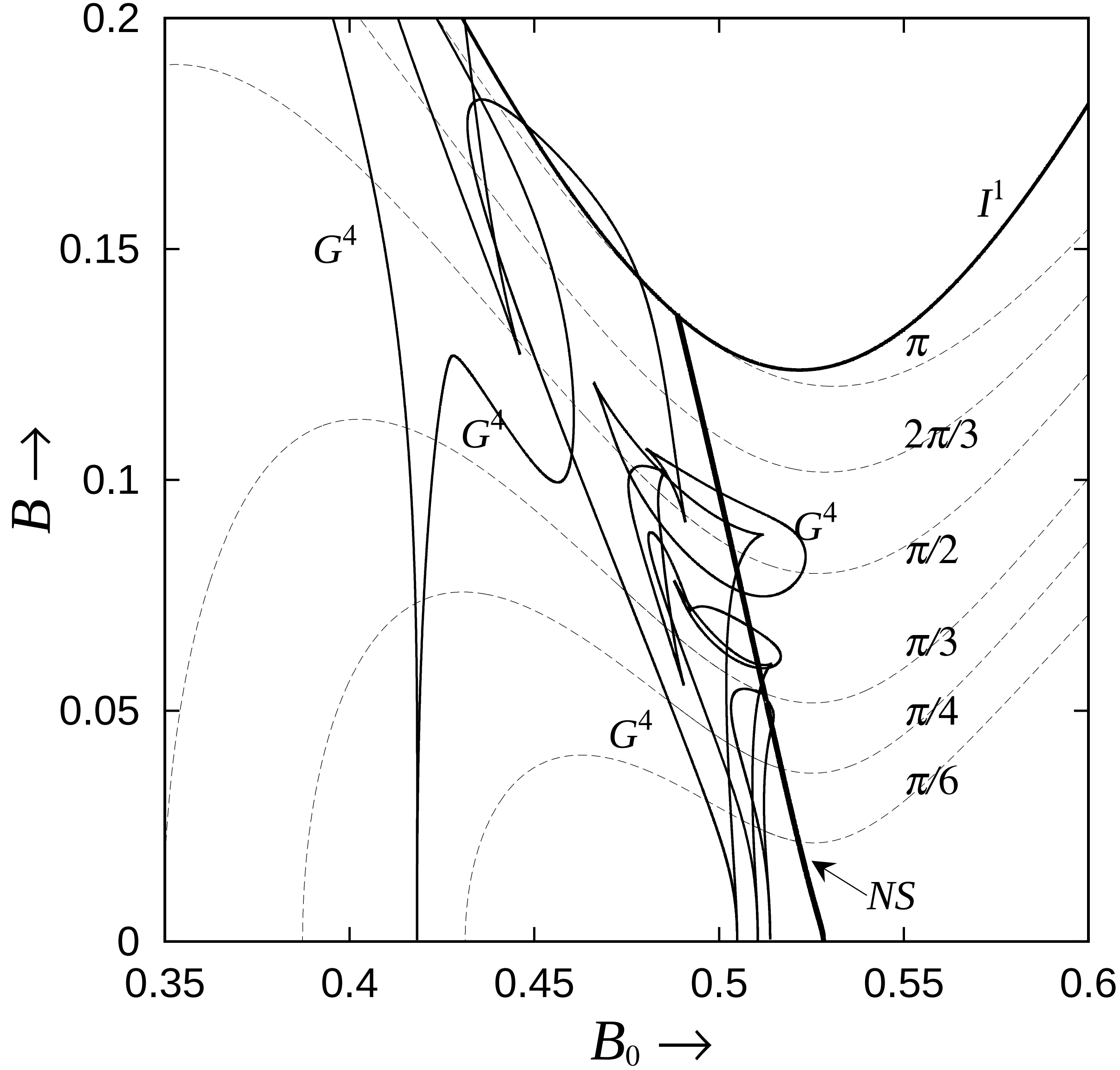
Forced BVP oscillator

Another neuron model:

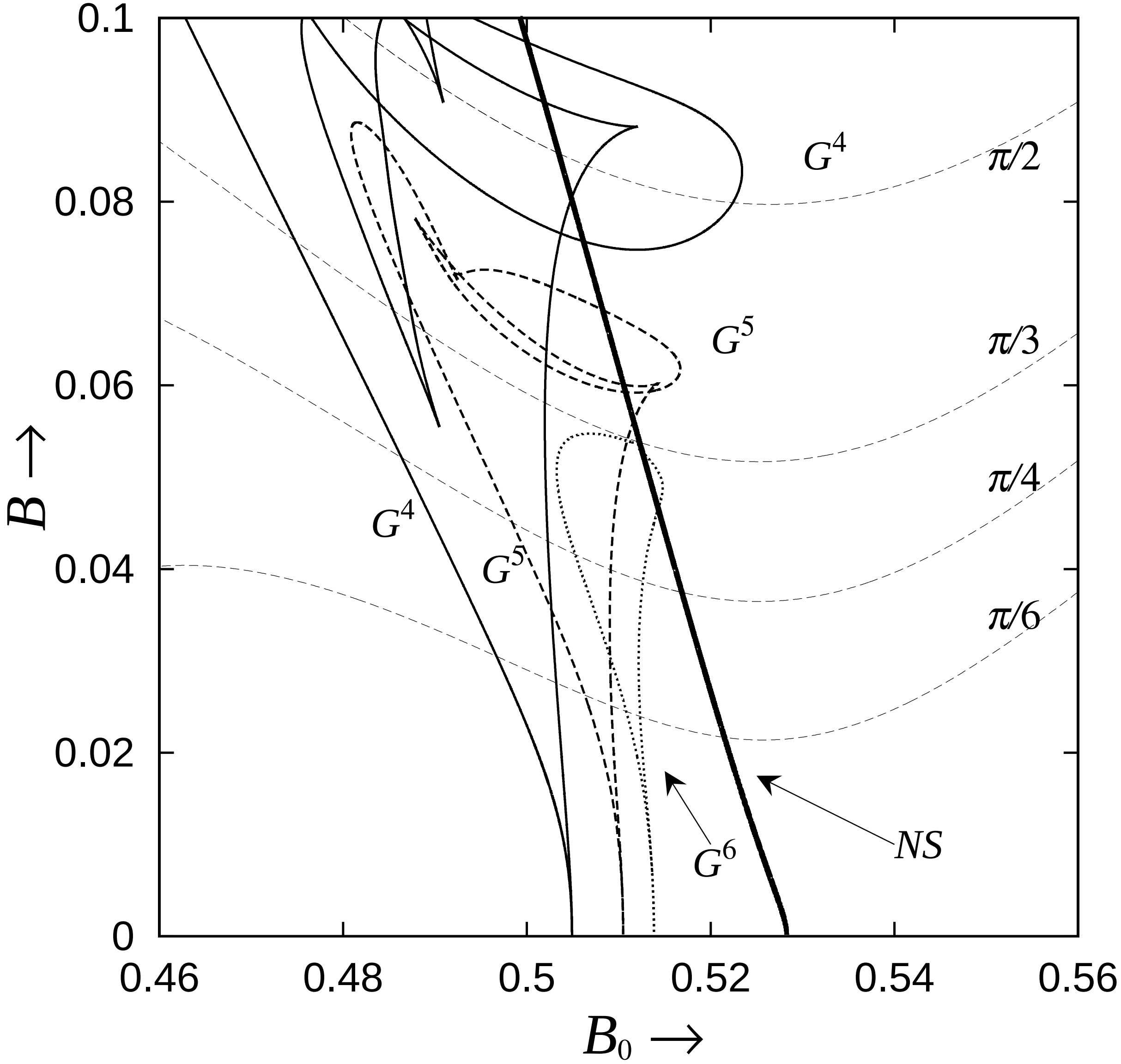
$$\begin{cases} \dot{x} = y + 1.6x - x^3 \\ \dot{y} = -x - 0.77y + B_0 + B\cos t \end{cases}$$

- quasi-periodic solutions
- chaotic motion
- frequency entrainment

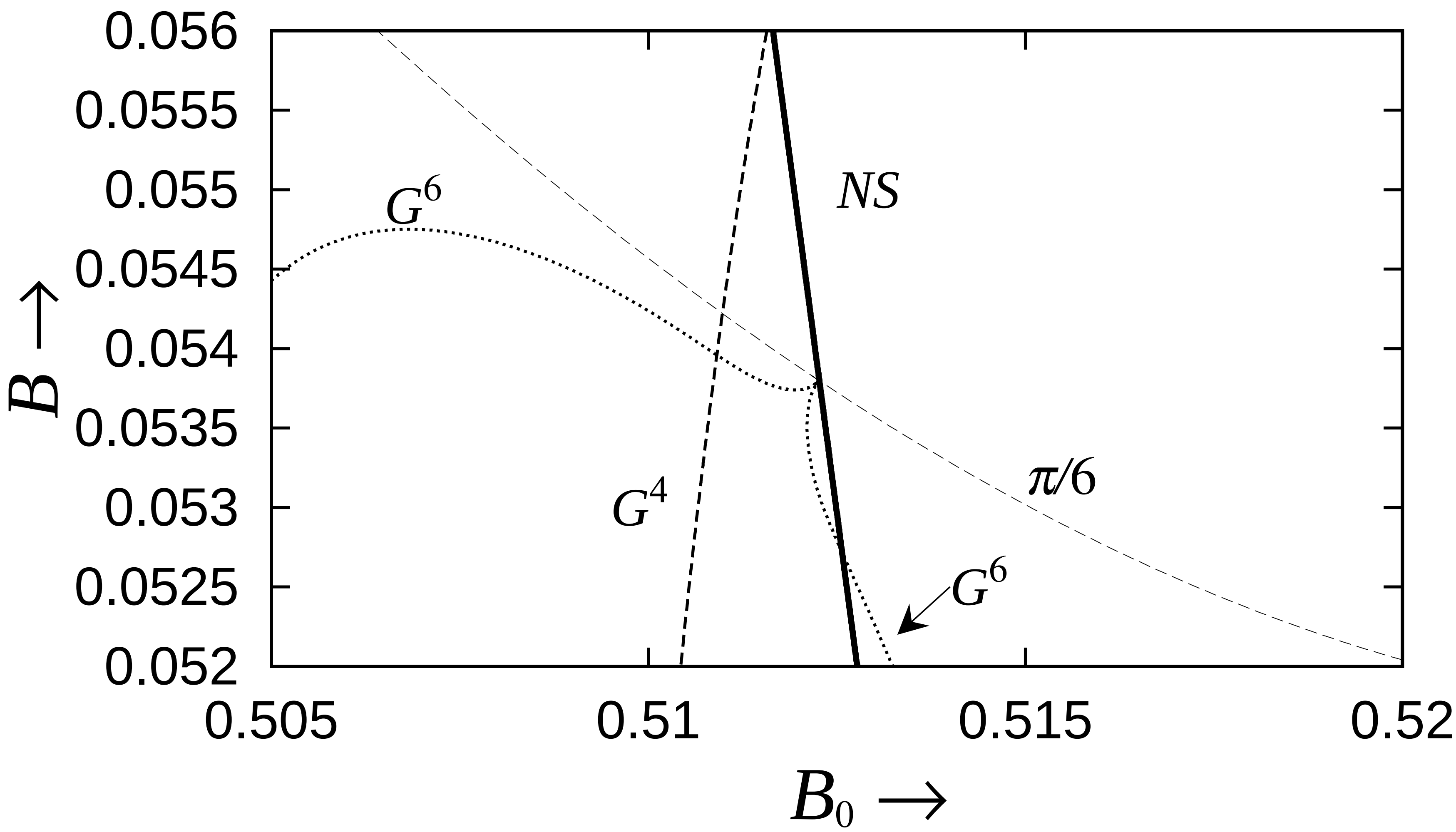
Bifurcation diagram



Bifurcation diagram2



Bifurcation diagram



Conclusions

Calculation of isocline with fixedargument for the fixedpoint

1. several isoclines corresponding to the special value of arguments
2. stability and instantaneous phase
3. a new calculation method of NS bifurcation
4. cross points of a cusp in a periodic entrainment region and the NS bifurcation curve