Calculation of the Isocline for the Fixed Point with a Specified Argument of Complex Multipliers

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Background

Analysis of nonlinear dynamical system described by ODE or DE

Equilibria or fixed/periodic points:
  - location
  - stability $\rightarrow$ bifurcation

Calculation of periodic points: solving boundary value problem
  - shooting method; Newton’s method
  - collocation method
  - automatic differentiation
Proposal

1. Calculation of isocline: a fixed argument of the complex multipliers
   - simple shooting algorithm
   - classify topological property of periodic points

2. an application of calculation: calculation of Neimark-Sacker bifurcation.
Description of the Problem

A discrete map $T$:

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^n; \quad u \mapsto T(u), \quad u \in \mathbb{R}^n$$

- Parameters: $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}$
- $T$ is $C^\infty$ for $u$, $\lambda_1$, and $\lambda_2$.

The fixed point $u_0$:

$$T(u_0) = u_0.$$
Complex multipliers

A pair of complex conjugate multipliers:

\[ \mu, \bar{\mu} = re^{\pm i\theta} = r(\cos \theta \pm i \sin \theta). \]

\(r\): the radius, \(\theta\): the argument.

\(r\) and \(\theta\) determine stability and angular velocity of the local linear space about \(u_0\).
Complex multipliers

limit cycle

quasi-periodic

NS Bifurcation

T. Ueta, for ISCAS 2001 – p.6/??
Calculation of the fixed point

$J$: the Jacobian matrix of $T$ at $u_0$

The characteristic equation:

$$\chi = \text{det}[J - re^{i\theta}] = \mathbb{R}\chi + i\mathbb{S}\chi = 0$$

Condition of the fixed point with $\theta$

$$F = \begin{pmatrix} T(u) - u \\ \mathbb{R}\chi \\ \mathbb{S}\chi \end{pmatrix} = 0$$

Variables: $(u, \lambda_1, r) \in \mathbb{R}^{n+2}$. 
Jacobian matrix

Then the Jacobian matrix for Newton’s method:

\[
D F = \begin{pmatrix}
\frac{\partial T}{\partial u_0} - I_n & \frac{\partial T}{\partial \lambda_1} & \frac{\partial T}{\partial r} \\
\frac{\partial \mathcal{R} \chi}{\partial u_0} & \frac{\partial \mathcal{R} \chi}{\partial \lambda_1} & \frac{\partial \mathcal{R} \chi}{\partial r} \\
\frac{\partial \mathcal{S} \chi}{\partial u_0} & \frac{\partial \mathcal{S} \chi}{\partial \lambda_1} & \frac{\partial \mathcal{S} \chi}{\partial r}
\end{pmatrix}
\]
The variational equation of $T$

For ODE:

$$\frac{d}{dt} \frac{\partial \varphi}{\partial x_0} = \frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial x_0}$$

$$\frac{d}{dt} \frac{\partial \varphi}{\partial \lambda_1} = \frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial x_0} + \frac{\partial f}{\partial \lambda_1}$$

can be obtained by numerical numerical integration like Runge-Kutta method.
**Isocline**

For the fixed $\theta \in [0, \pi]$, $(u_0, \lambda_1, r)$ forms an isocline in the parameter plane $\lambda_1 - \lambda_2$.

$r = 0$

$r = 1$

$r = \infty$

Neimark-Sacker

Stable

Unstable

T. Ueta, for ISCAS 2001 – p.10/?
Properties

- $\theta = 0$ or $\theta = \pi$: critical line between real and complex multipliers. (nodal or rotational linear space about $u_0$.)
- any rational ratio with $\pi$ of $\theta$: deeply related to the NS bifurcation and frequency entrainment regions.
Calculation of NS bifurcation

Exchange variables and parameters: $r \Leftrightarrow \theta$
Let $r = 1$, then solve

$$F = \begin{pmatrix} T(u) - u \\ \Re \chi \\ \Im \chi \end{pmatrix} = 0$$

for $(u, \lambda_1, \lambda_2) \in \mathbb{R}^{n+2}$
The Jacobian matrix

\[
DF = \begin{pmatrix}
\frac{\partial T}{\partial u_0} - I & \frac{\partial T}{\partial \lambda_1} & \frac{\partial T}{\partial \lambda_2} \\
\frac{\partial R\chi}{\partial u_0} & \frac{\partial R\chi}{\partial \lambda_1} & \frac{\partial R\chi}{\partial \lambda_2} \\
\frac{\partial S\chi}{\partial u_0} & \frac{\partial S\chi}{\partial \lambda_1} & \frac{\partial S\chi}{\partial \lambda_2}
\end{pmatrix}
\]
Features

Neimark-Sacker bifurcation — explicitly parameterized by $\theta$.

- Very simple formulation to calculate NS bifurcation.
- Codimension-two bifurcation parameter values can be obtained, $\theta = 0$ and $\theta = \pi$.
- No predictor-corrector method is needed.
- Not sensitive for stability of the periodic point.
Example 1

A Discrete Chaotic Map

\[
\begin{align*}
x_1(k + 1) &= x_2(k) + ax_1(k) \\
x_2(k + 1) &= x_1^2(k) + b
\end{align*}
\]

\(k = 0, 1, 2, \cdots\)

The characteristic equation:

\[
\Re \chi = r^2(\cos^2 \theta - \sin^2 \theta) - r \cos \theta \text{ tr } X + \text{ det } X
\]

\[
\Im \chi = 2r \cos \theta - \text{ tr } X
\]

\(X\): the fundamental solution matrix of the variational equation.
Bifurcation diagram
Bifurcation diagram
NS bifurcation: \( b = 0.5a - 0.75 \)
Period-6, $\theta = \pi/3$

\[ a = 1.0, \quad b = -0.25, \quad \mathbf{u}_0 = (-0.5, 0.0). \]
period-5, $\theta = \frac{2\pi}{5}$

$a = 0.618033, \ b = -0.440983$, 

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A couple of neurons

A system described by ODE:

\[
\begin{align*}
\dot{x}_1 &= 10 f(x_1) - 10 f(x_2) - 2.5 + B \cos \omega t \\
\dot{x}_2 &= 10 f(x_1) + 2 f(x_2) - 9.0
\end{align*}
\]

\[
f(x) = \frac{1}{1 + e^{-x}}
\]

- quasi-periodic solutions
- chaotic motion
- frequency entrainment
Bifurcation diagram
Period-5, $\theta = \frac{2\pi}{5}$
Period-6, $\theta = 2\pi/7$. 
Forced BVP oscillator

Another neuron model:

\[\begin{align*}
\dot{x} &= y + 1.6x - x^3 \\
\dot{y} &= -x - 0.77y + B_0 + B \cos t
\end{align*}\]

- quasi-periodic solutions
- chaotic motion
- frequency entrainment
Bifurcation diagram
Bifurcation diagram 2
Conclusions

Calculation of isocline with fixed argument for the fixed point

1. several isoclines corresponding to the special value of arguments
2. stability and instantaneous phase
3. a new calculation method of NS bifurcation
4. cross points of a cusp in a periodic entrainment region and the NS bifurcation curve