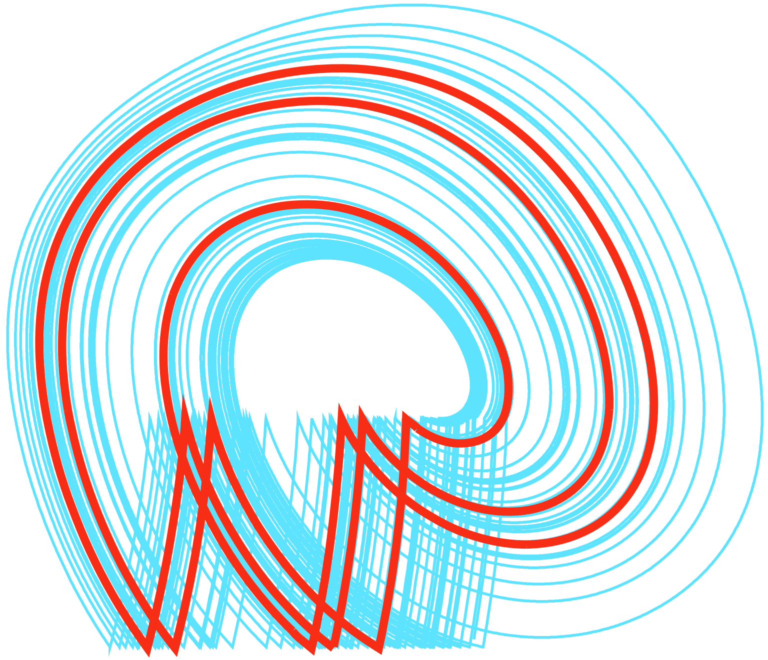
離散切替力学系の非線形現象とその解析



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本発表の位置づけ

ハイブリッドシステム…連続力学系と離散力学系の混在，相互作用のある系

- ✎ 力学系 (特に電気回路) からみたハイブリッドシステムの分類
- ✎ ハイブリッドシステムの非線形現象・特有の分岐現象
- ✎ カオス制御への応用

非線形力学系としての Hybrid Systems

Guckenheimer and Johnson, Planer Hybrid Systems, Hybrid Systems II LNCS999, 202/255, Springer, 1995.

しかし…非線形問題としては 20 年以上前から精力的に研究されていたと…思う.

- ✎ 京都大学林研 1960 年代?
- ✎ 法政大学斎藤研 1980 年
- ✎ U. C. Berkeley, Prof. Chua 1986 年

力学屋の興味

- 特異点 (平衡点，固定点，周期点), 周期解
  - 個数の増減
  - 安定性
  - 分岐とカオス
- 1990 年代に入ってカオス制御のムーブメントが巻き起るまでは，特に制御問題としての意識はなかった

電気回路でのハイブリッドシステム

- リレーやスイッチ
  - **periodic switch:** 非自律的 (他動的)
  - **threshold switch:** 自律的
- 回路特性における **break points**
  - 区分的 (線形・非線形) に定義される力学系
  - ヒステリシス
- 不連続波形の印加: 非自律系

分類

1. **Threshold impulse stimuli:** 状態に依存して方程式が不連続に切替られる系 (断続系 **(interrupted dynamical system):** ある条件を満たしたときにインパルスが入る系など
2. **Periodic impulse stimuli** 時刻に依存する断続系**:** インパルス周期外力が印加される系など
3. **Autonomous:**状態に依存した断続系だが，**break point** で軌道は連続であるもの**:** 区分線形系など
4. **Nonautonomous:**時刻に依存した断続系だが，**break point** で軌道は連続である場合**:** 矩形波や全波整流波を加えた系など

われわれの果たした役割

1994–1995: **Periodic impulse stimuli** による周期解の分岐について，計算方法を提案・完成

1995–1997: 非自律的断続系の分岐計算を完成

1997–1998: 状態に依存した自律系断続系での周期解の **Poincaré** 写像における特性乗数の計算方法を考案

1999–present: カオス制御への応用

ちょっとした主張

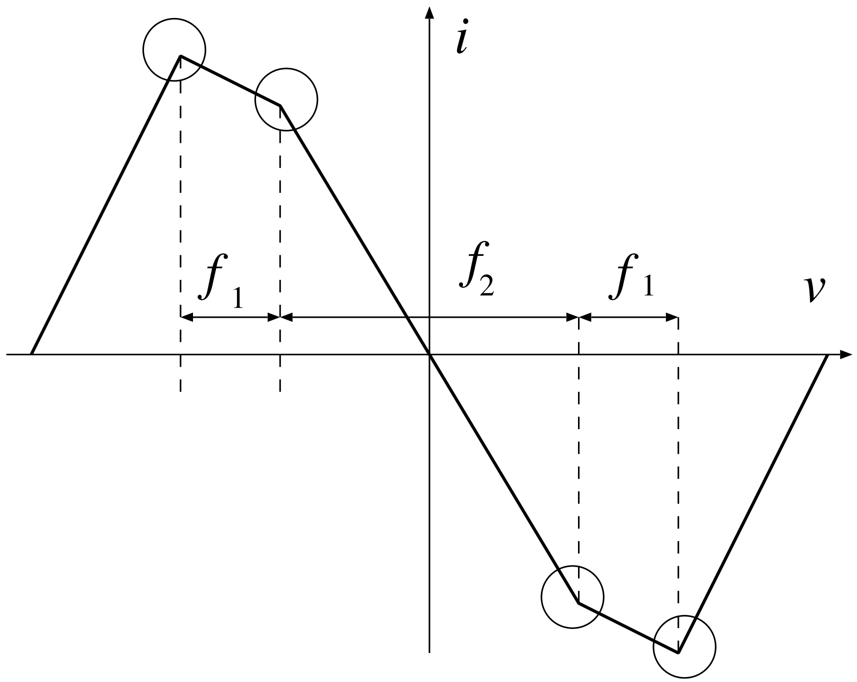
T. Ueta and H. Kawakami,“Composite dynamical system for controlling chaos,” IEICE Trans. Fundamentals, E78-A(6):708-714, June 1995.

Composite dynamical system

- ✎ Poincaré 写像による離散化
- ✎ 離散系においてフィードバック制御設計
- ✎ 連続系にはパラメータ摂動で制御量を還元

連続系から導かれた離散系における動的スキームが，もとの連続系の動作に影響を与える— **Hybrid system**

Chua 回路

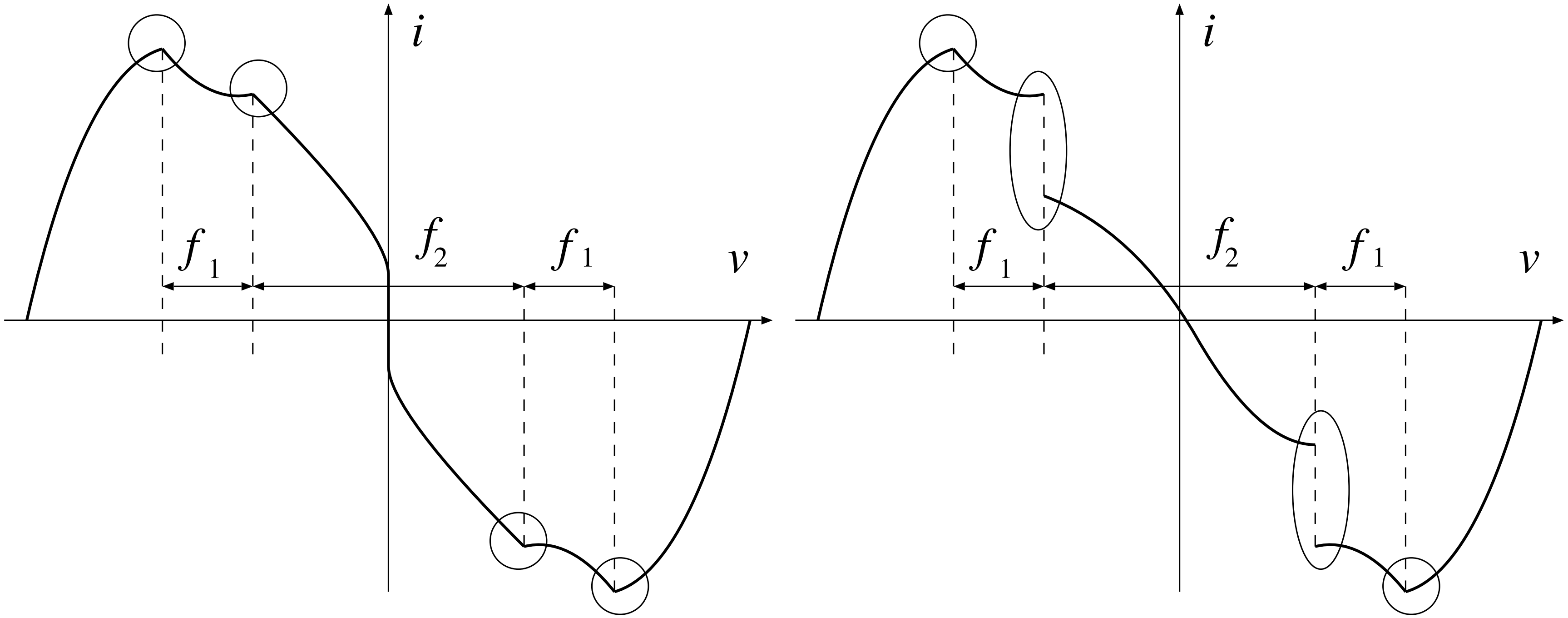


Analysis for the vector field of piecewise-linear equation ⇒ can be achieved rigorously

- ✎ **Poincaré mapping**
- ✎ **Special characteristics of Jacobian matrix**

General piecewise-defined equations

⇒ **Not smooth characteristics**



- ✎ **impossible to obtain an exact solution**
- ✎ **required some technical preparations to construct the Poincaré mapping**

Piecewise-Defined Differential Equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{\lambda}) \quad i = 0, 1, 2 \cdots, m - 1$$

$t \in \boldsymbol{R}$, $\boldsymbol{x} \in \mathrm{R}^{\mathrm{n}}$, $\boldsymbol{\lambda} \in \boldsymbol{R}^r$, $\boldsymbol{f}_i$ is C^∞-class map

Piecewise-defined differential equations

A solution:

$$\boldsymbol{x}_i(t) = \boldsymbol{\varphi}_i(t, \boldsymbol{x}_{i0}) \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0}$$

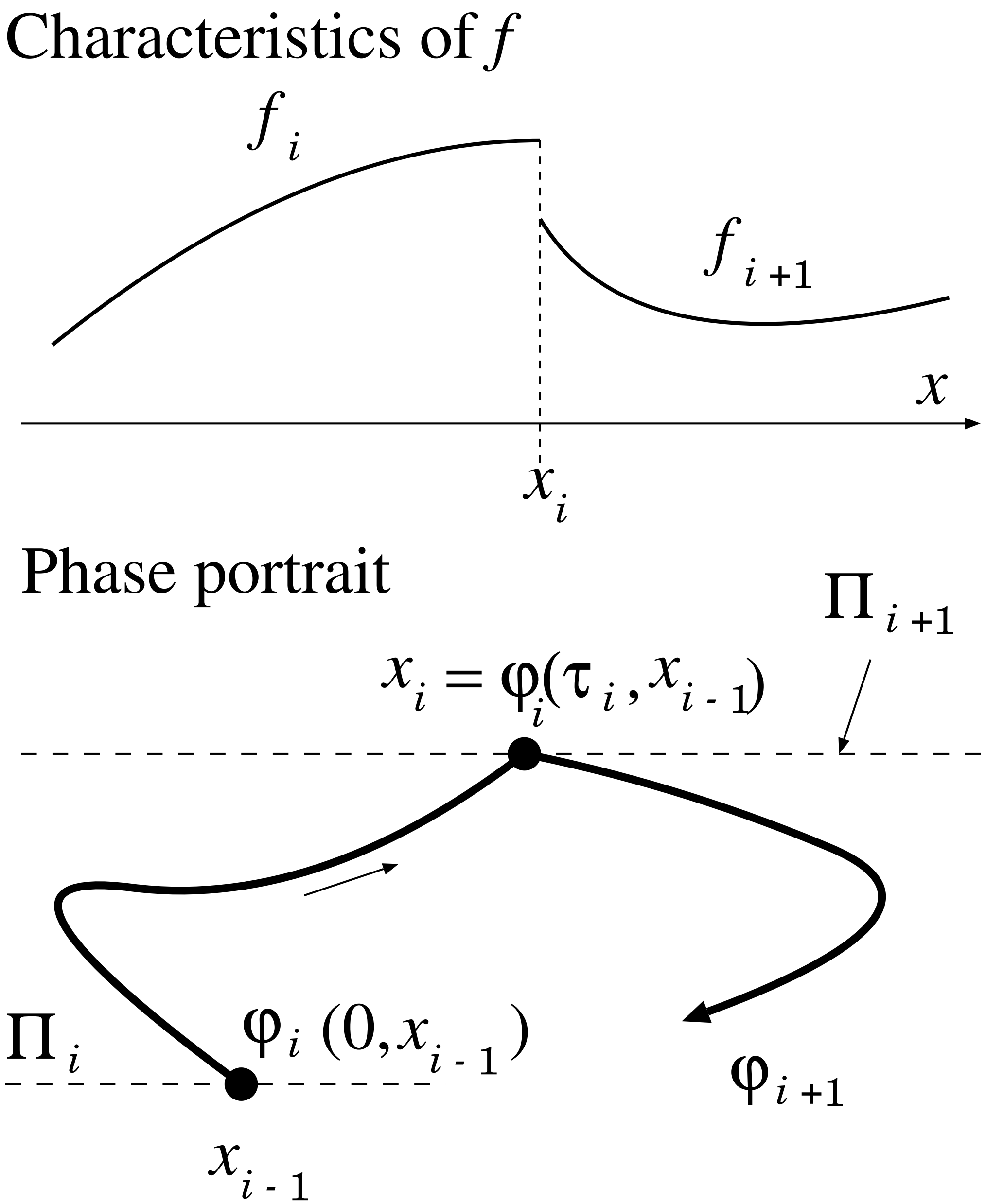
$\boldsymbol{x}_0$ **: an arbitrary initial value.**

Condition

Poincaré section must be located at every break point defined by $q_i(x_i)$**:**

$$\Pi_i = \{\boldsymbol{x}_i \in \boldsymbol{R}^n \,|\, q_i(\boldsymbol{x}_i) = 0\}$$
$$i = 0, 1, 2, \cdots, m-1$$

Connection



Characteristics of f
f_i
f_{i+1}
x
x_i
Phase portrait
Π_{i+1}
$x_i = \varphi_i(\tau_i, x_{i-1})$
Π_i
$\varphi_i(0, x_{i-1})$
φ_{i+1}
x_{i-1}

Connection

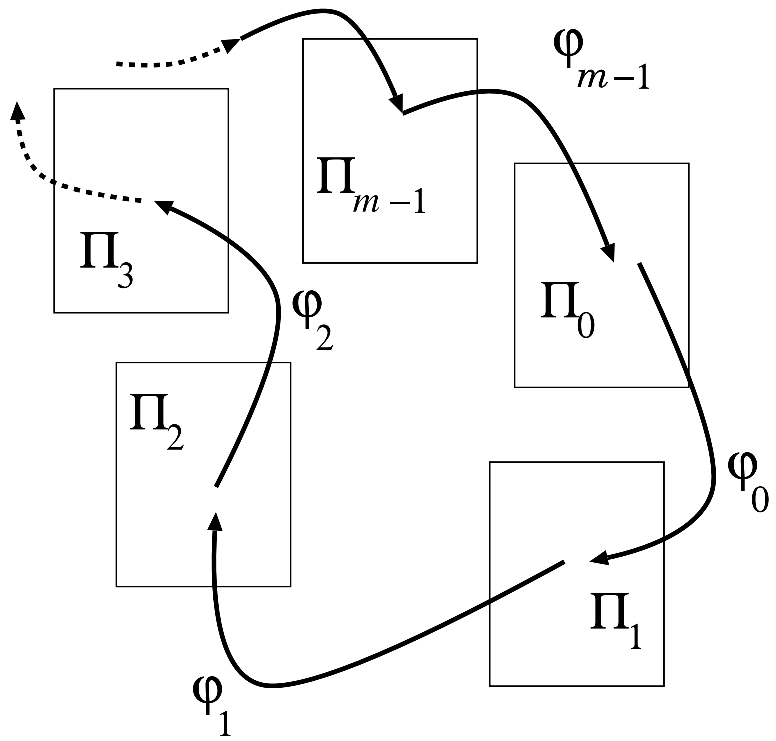
A flow until $i+2$-th section:

$$\boldsymbol{x}_{i+1}(t) = \boldsymbol{\varphi}_{i+1}(t, \boldsymbol{x}_i)$$

where

$$\boldsymbol{x}_{i+1}(0) = \boldsymbol{x}_i = \boldsymbol{\varphi}_{i+1}(0, \boldsymbol{x}_i) = \boldsymbol{\varphi}_i(\tau_i, \boldsymbol{x}_{i-1})$$

A periodic solution (limit cycle)



$$\boldsymbol{x}_0 = \boldsymbol{x}_m = \boldsymbol{\varphi}_{m-1}(\tau_m, \boldsymbol{x}_{m-1})$$

- ✎ **The limit cycle hits all Poincaré sections.**
- ✎ **The limit cycle is continuous but not differentiable for all states.**

Definition of local mappings

$$\begin{array}{llcl}
T_0: & \Pi_0 & \longrightarrow & \Pi_1 \\
 & \boldsymbol{x}_0 & \longmapsto & \boldsymbol{x}_1 = \boldsymbol{x}(\tau_0) = \boldsymbol{\varphi}_0(\tau_0, \boldsymbol{x}_0) \\
T_1: & \Pi_1 & \longrightarrow & \Pi_2 \\
 & \boldsymbol{x}_1 & \longmapsto & \boldsymbol{x}_2 = \boldsymbol{x}(\tau_1) = \boldsymbol{\varphi}_1(\tau_1, \boldsymbol{x}_1) \\
 & & \vdots & \\
T_{m-1}: & \Pi_{m-1} & \longrightarrow & \Pi_0 \\
 & \boldsymbol{x}_{m-1} & \longmapsto & \boldsymbol{x}_0 = \boldsymbol{\varphi}_{m-1}(\tau_{m-1}, \boldsymbol{x}_{m-1})
\end{array}$$

period of the limit cycle

$$\tau = \sum_{i=0}^{m-1} \tau_i$$

Poincaré mapping (differentiable)

$$T = T_0 \circ T_1 \circ \cdots \circ T_{m-1}$$

Jacobian matrix of T

$$\left.\frac{\partial T}{\partial \boldsymbol{x}_0}\right|_{t=\tau} = \prod_{i=0}^{m-1} \left.\frac{\partial T_i}{\partial \boldsymbol{x}_i}\right|_{t=\tau_i}$$

where

$$\frac{\partial T_i}{\partial \boldsymbol{x}_i} = \frac{\partial \boldsymbol{x}_{i+1}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{x}_i} + \frac{\partial \boldsymbol{\varphi}_i}{\partial t}\frac{\partial \tau_i}{\partial \boldsymbol{x}_i} = \frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{x}_i} + \boldsymbol{f}_0 \frac{\partial \tau_i}{\partial \boldsymbol{x}_i}$$

Equations of Poincaré sections are also differentiable:

$$q_{i+1}(\boldsymbol{x}_i) = q_{i+1}(\boldsymbol{\varphi}_i, \boldsymbol{x}_0) = 0$$

$$\Downarrow$$

$$\frac{\partial q_{i+1}}{\partial \boldsymbol{x}_i}\left(\frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{x}_i} + \boldsymbol{f}_i \frac{\partial \tau_i}{\partial \boldsymbol{x}_i}\right) = \mathbf{0}$$

where $q_m = q_0$**.**

$$\frac{\partial T_i}{\partial \boldsymbol{x}_i} = \frac{\partial \boldsymbol{x}_{i+1}}{\partial \boldsymbol{x}_i} = \left[\boldsymbol{I}_n - \frac{1}{\frac{\partial q_{i+1}}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}_i} \boldsymbol{f}_i \cdot \frac{\partial q_{i+1}}{\partial \boldsymbol{x}} \right] \frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{x}_i}$$

$\partial \boldsymbol{\varphi}_i / \partial \boldsymbol{x}_0$ can be obtained by solving the following variational equations from $t = 0$ to $t = \tau_i$.

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{x}_i}\right) = \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}}(t, \boldsymbol{\varphi}(t, \boldsymbol{x}_i)) \left(\frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{x}_i}\right)$$

$$\left.\frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{x}_i}\right|_{t=0} = \boldsymbol{I}_n \qquad i = 0, 1, 2, \cdots, m-1.$$

A local coordinate $\boldsymbol{u} \in \Sigma_0 \subset \boldsymbol{R}^{n-1}$

with projection p and embedding map p^{-1}

$$p : \Pi_0 \rightarrow \Sigma_0, \quad p^{-1} : \Sigma_0 \rightarrow \Pi_0$$

Poincaré mapping on the local coordinate:

$$T_\ell : \Sigma_0 \rightarrow \Sigma_0;\ \boldsymbol{u} \mapsto p \circ T \circ p^{-1}(\boldsymbol{u})$$

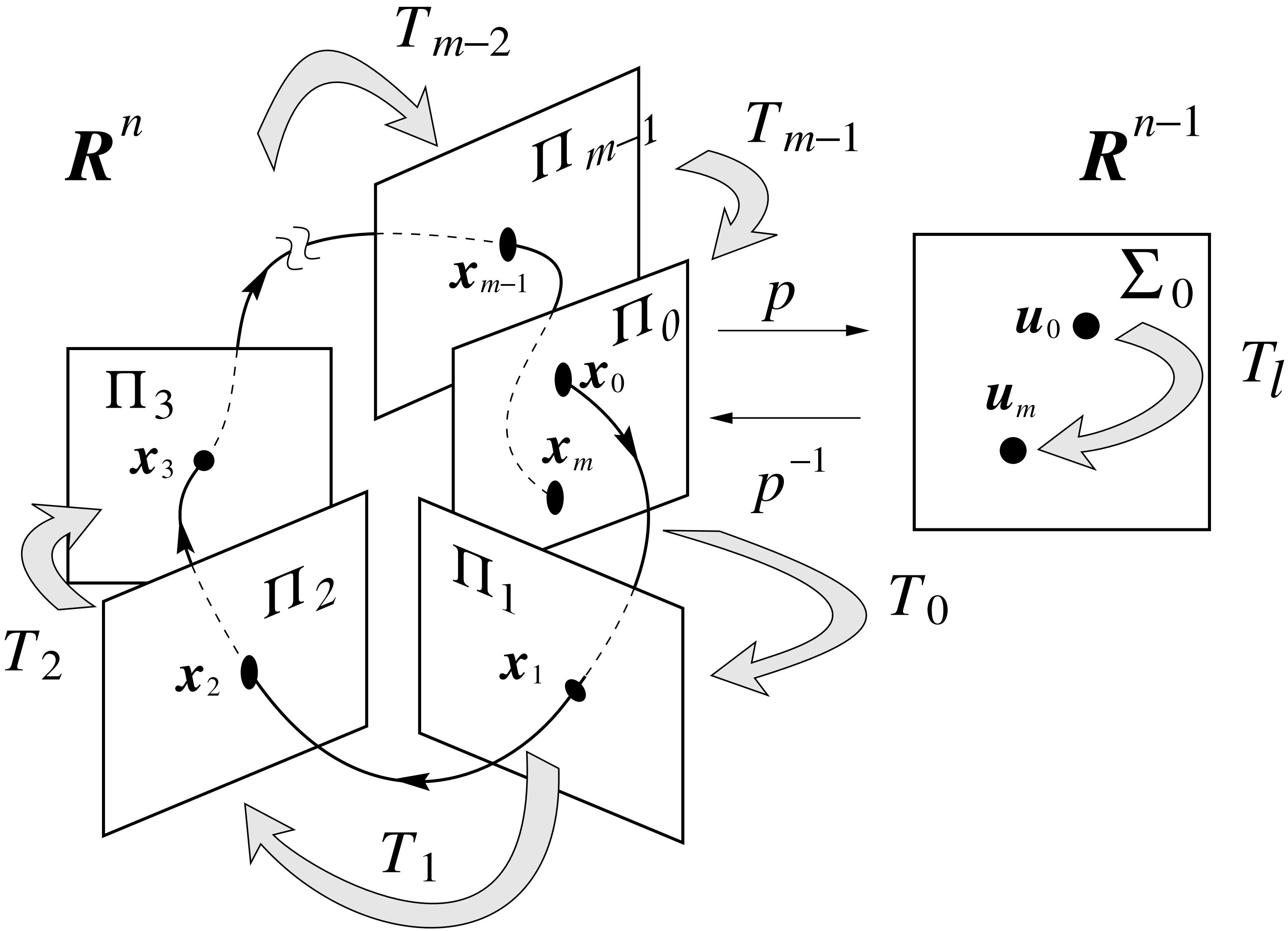
A fixed point of the Poincaré mapping is obtained by solving the following equation.

$$T_\ell(\boldsymbol{u}) - \boldsymbol{u} = \mathbf{0}$$

The Jacobian matrix required in Newton's method is given by:

$$
\begin{aligned}
\frac{\partial T_\ell}{\partial \boldsymbol{u}_0} &= DT_\ell(\boldsymbol{u}_0) = \frac{\partial p}{\partial \boldsymbol{x}} \frac{\partial T}{\partial \boldsymbol{x}_0} \frac{\partial p^{-1}}{\partial \boldsymbol{u}} \\
&= \frac{\partial p}{\partial \boldsymbol{x}} \left[\boldsymbol{I}_n - \frac{1}{\frac{\partial q_0}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}_0} \boldsymbol{f}_0 \cdot \frac{\partial q_0}{\partial \boldsymbol{x}} \right] \frac{\partial \boldsymbol{\varphi}_0}{\partial \boldsymbol{x}_0} \frac{\partial p^{-1}}{\partial \boldsymbol{u}}
\end{aligned}
$$

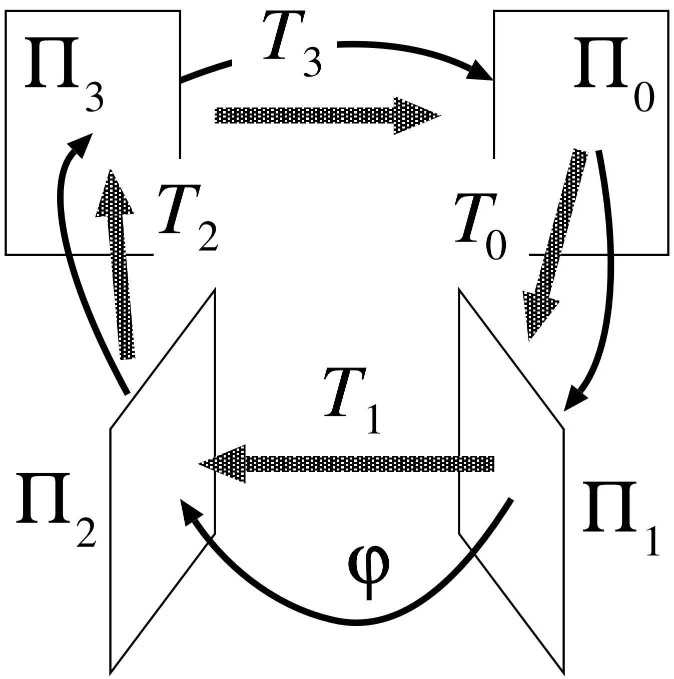
Composite of T_k and local coordinate Σ



Why do we prepare the local coordinate ??

In case: f is smooth

Multiple Poincaré section for a limit cycle:



$$\frac{\partial \varphi}{\partial \boldsymbol{x}_0} = \frac{\partial T_0}{\partial \boldsymbol{x}_0} \cdot \frac{\partial T_1}{\partial \boldsymbol{x}_1} \cdot \frac{\partial T_2}{\partial \boldsymbol{x}_2} \cdot \frac{\partial T_3}{\partial \boldsymbol{x}_3}$$

$$= \frac{\partial \varphi_0}{\partial \boldsymbol{x}_0} \cdot \frac{\partial \varphi_1}{\partial \boldsymbol{x}_1} \cdot \frac{\partial \varphi_2}{\partial \boldsymbol{x}_2} \cdot \frac{\partial \varphi_3}{\partial \boldsymbol{x}_3}$$

Multiple shooting method for smooth f_k

The roots of the characteristic equation

$$\left|\frac{\partial \varphi}{\partial \boldsymbol{x}_0} - \mu I_n\right| = 0$$

are coincident with correct eigenvalues (multipliers) of the fixed point of the Poincaré mapping

In case: f is smooth

$$\frac{\partial T_\ell}{\partial \boldsymbol{u}_0} = DT_\ell(\boldsymbol{u}_0) = \frac{\partial p}{\partial \boldsymbol{x}} \frac{\partial T}{\partial \boldsymbol{x}_0} \frac{\partial p^{-1}}{\partial \boldsymbol{u}}$$

where

$$\left. \frac{\partial T}{\partial \boldsymbol{x}_0} \right|_{t=\tau} = \prod_{i=0}^{m-1} \left. \frac{\partial T_i}{\partial \boldsymbol{x}_i} \right|_{t=\tau_i}$$

$$\frac{\partial T_i}{\partial \boldsymbol{x}_i} = \frac{\partial \boldsymbol{x}_{i+1}}{\partial \boldsymbol{x}_i} = \left[\boldsymbol{I}_n - \frac{1}{\frac{\partial q_{i+1}}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}_i} \boldsymbol{f}_i \cdot \frac{\partial q_{i+1}}{\partial \boldsymbol{x}} \right] \frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{x}_i}$$

$n-1$ roots of the characteristic equation

$$\left|\frac{\partial T_\ell}{\partial \boldsymbol{u}_0} - \mu I_{n-1}\right| = 0$$ **are coincident with roots of**

$$\left|\frac{\partial \varphi}{\partial \boldsymbol{x}_0} - \mu I_n\right| = 0$$ **except** $\mu = 1$**.**

$$\Downarrow$$

We can use $\left|\frac{\partial \varphi}{\partial \boldsymbol{x}_0} - \mu I_n\right| = 0$ **instead of**

$$\left|\frac{\partial T_\ell}{\partial \boldsymbol{u}_0} - \mu I_{n-1}\right| = 0$$

Jacobian matrices become more simple. $\Rightarrow$ **A number of variational equations can be reduced.**

区分線形系でもこの方程式の根と，特性乗数は一致

In case: f is not smooth

$$\chi(\mu) = |DT - \mu \boldsymbol{I}_n| = 0 \quad \textbf{or} \quad \left| \frac{\partial \varphi}{\partial \boldsymbol{x}_0} - \mu I_n \right| = 0$$

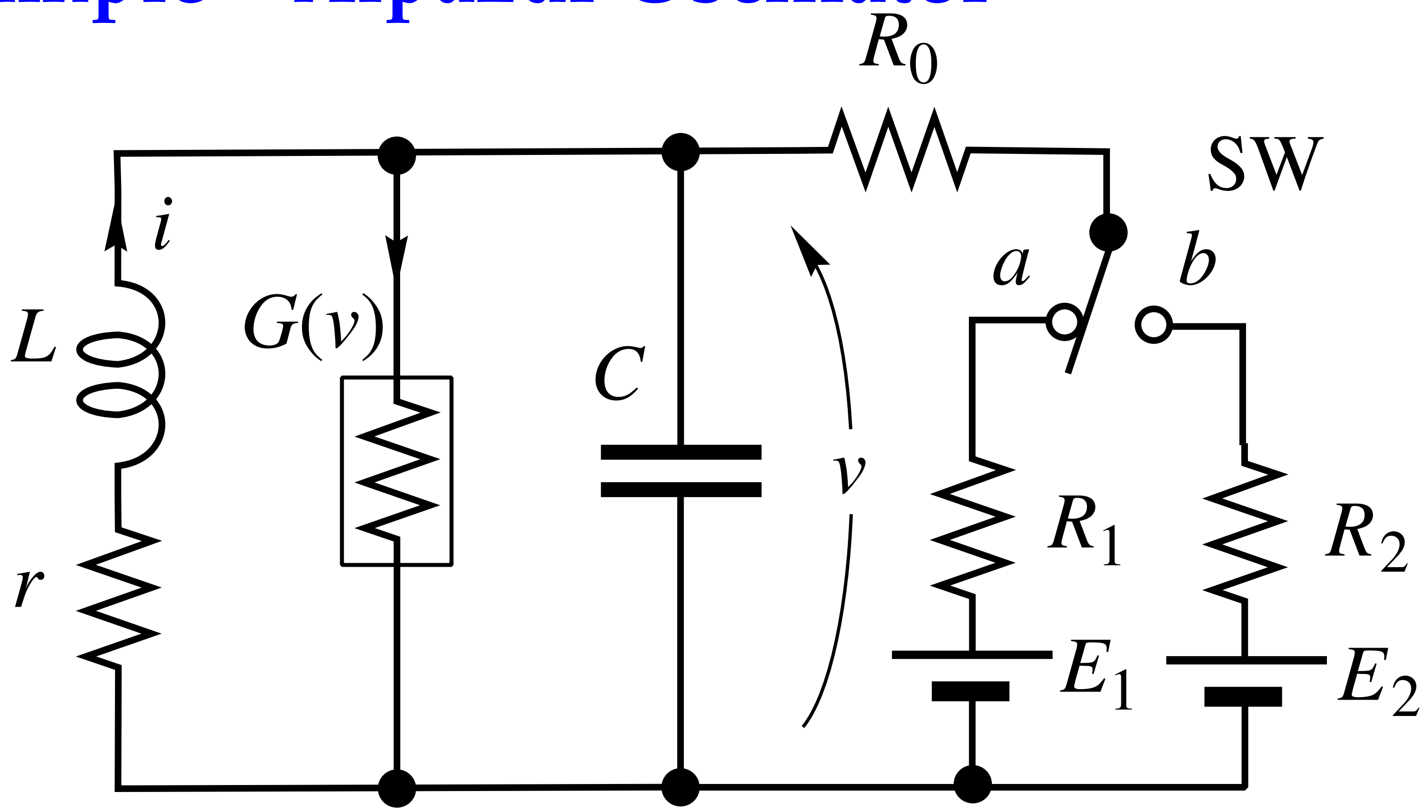
has no correct eigenvalues. We must use

$$\chi_\ell(\mu) = |DT_\ell - \mu \boldsymbol{I}_{n-1}| = 0$$

- ✎ **The vector field at the Poincaré section is discontinuous.**
- ✎ **In a piecewise linear system,** $\left| \frac{\partial \varphi}{\partial \boldsymbol{x}_0} - \mu I_n \right| = 0$ **return correct eigenvalues. (unconfirmed)**

Newton's method:

Example—Alpazur Oscillator

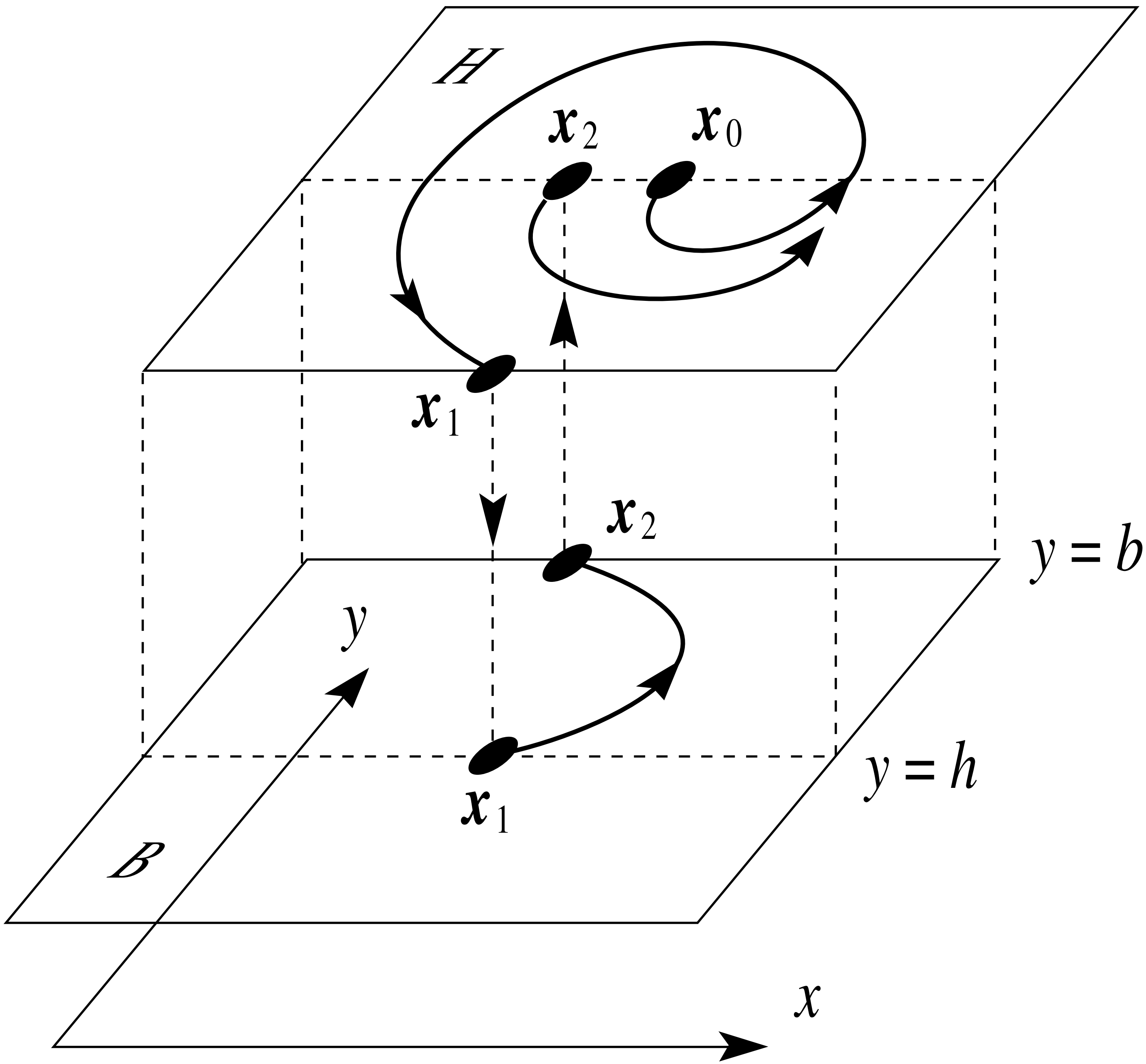


Example—Alpazur Oscillator

H. Kawakami and R. Lozi. "Switched Dynamical Systems — dynamical of a class of circuits with switch —," Proc. RIMS Conf. "Structure and Bifurcations of Dynamical Systems," ed. by S. Ushiki, World Scientific, pp.39–58, 1992.

- ✎ **An nonlinear resistor $G(v) = v - v^3/3$.**
- ✎ **A switch SW depends the states of the system $\Rightarrow$ Hysteresis properties**

Hysteresis characteristics



Alpazur equation

$$\textbf{SW:}a \begin{cases} \dfrac{dx}{dt} = -rx - y & = f_0 \\ \dfrac{dy}{dt} = x + (1 - g_1)y - \dfrac{1}{3}y^3 + B_1 & = g_0 \end{cases}$$

$$\textbf{SW:}b \begin{cases} \dfrac{dx}{dt} = -rx - y & = f_1 \\ \dfrac{dy}{dt} = x + (1 - g_2)y - \dfrac{1}{3}y^3 + B_2 & = g_1 \end{cases}$$

$$H = \{(x,y) | \, y \geq h\}, \quad B = \{(x,y) | \, y \leq b\}$$

$$\partial H = \{(x,y) | \, y = h\}, \quad \partial B = \{(x,y) | \, y = b\}, b > h$$

$$H : x(t) = \varphi_0(t, x_0, y_0, \lambda_0, \lambda),\; y(t) = \phi_0(t, x_0, y_0, \lambda_0, \lambda)$$

$$B : x(t) = \varphi_1(t, x_1, y_1, \lambda_1, \lambda),\; y(t) = \phi_1(t, x_1, y_1, \lambda_1, \lambda)$$

$$\begin{aligned}
\Pi_0 &= \{\boldsymbol{u} \in B \,|\, q_0(x_1, y_1) = y - b = 0\} \\
\Pi_1 &= \{\boldsymbol{u} \in H \,|\, q_1(x_1, y_1) = y - h = 0\}
\end{aligned}$$

$$\begin{aligned}
T_0 : \;\; \Pi_0 &\longrightarrow \Pi_1 \\
x_0 &\longmapsto x_1 = \varphi_0(\tau_0(x_0, y_0, \lambda_0, \lambda), x_1, y_1, \lambda_0, \lambda) \\
y_0 &\longmapsto y_1 = h \\
T_1 : \;\; \Pi_1 &\longrightarrow \Pi_0 \\
x_1 &\longmapsto x_2 = \varphi_1(\tau_1(x_0, y_0, \lambda_1, \lambda), x_1, y_1, \lambda_1, \lambda) \\
y_1 &\longmapsto y_2 = b
\end{aligned}$$

$$T = T_0 \circ T_1$$

$$p: \qquad \Pi_0 \longrightarrow \Sigma_0; \;\; \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \;\; \mapsto \;\; u = x$$

$$p^{-1}: \;\; \Sigma_0 \longrightarrow \Pi_0; \;\; u = x \qquad\qquad \mapsto \;\; \boldsymbol{x} = \begin{bmatrix} x \\ b \end{bmatrix}$$

The Jacobian matrix of the Poincaré mapping:

$$\begin{aligned} DT_\ell(\boldsymbol{u}) &= \frac{\partial p}{\partial \boldsymbol{x}} \frac{\partial T}{\partial \boldsymbol{u}_0} \frac{\partial p^{-1}}{\partial \boldsymbol{u}} \\ &= \left(\frac{\partial \varphi_0}{\partial x_0} - \frac{f_0}{g_0} \frac{\partial \phi_0}{\partial x_0} \right) \left(\frac{\partial \varphi_1}{\partial x_1} - \frac{f_1}{g_1} \frac{\partial \phi_1}{\partial x_1} \right) \end{aligned}$$

Global bifurcations: a special property of the system with non-smooth characteristics

$$\Downarrow$$

An orbit trapped into a singular orbit going to the pseudo-equilibrium

(the orbit touches the boundary tangentially)

$$T_\ell^m(\boldsymbol{x}_0) - \boldsymbol{x}_0 = 0, \quad y\big|_{t=\tau_i} = 0, \quad \left.\frac{dy}{dt}\right|_{t=\tau_i} = 0$$

例 1

東北大学大学院工学研究科 江村・玄研究室

- ✎ 受動 1 脚走行ロボットのモデル化
- ✎ 衝突をなくすための条件式を導出 (モードの正常な切替え条件も)
- ✎ 上記を満たすための制御入力を導出
- ✎ シミュレーションの結果，準周期軌道が出現
- ✎ ある周期に安定化するような制御則を考案 → **1, 2** 周期への安定化成功

周期解の計算にわれわれの結果を適用

例 2

カオス制御 — External force control

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{K}(\boldsymbol{x} - \boldsymbol{x}^*(t))$$

方針:

1. カオス中の不安定軌道を高精度に計算可能
2. 不安定軌道の情報をメモリに蓄え，垂れ流しフィードバックする
3. f が区分線形・区分非線形でも制御は可能

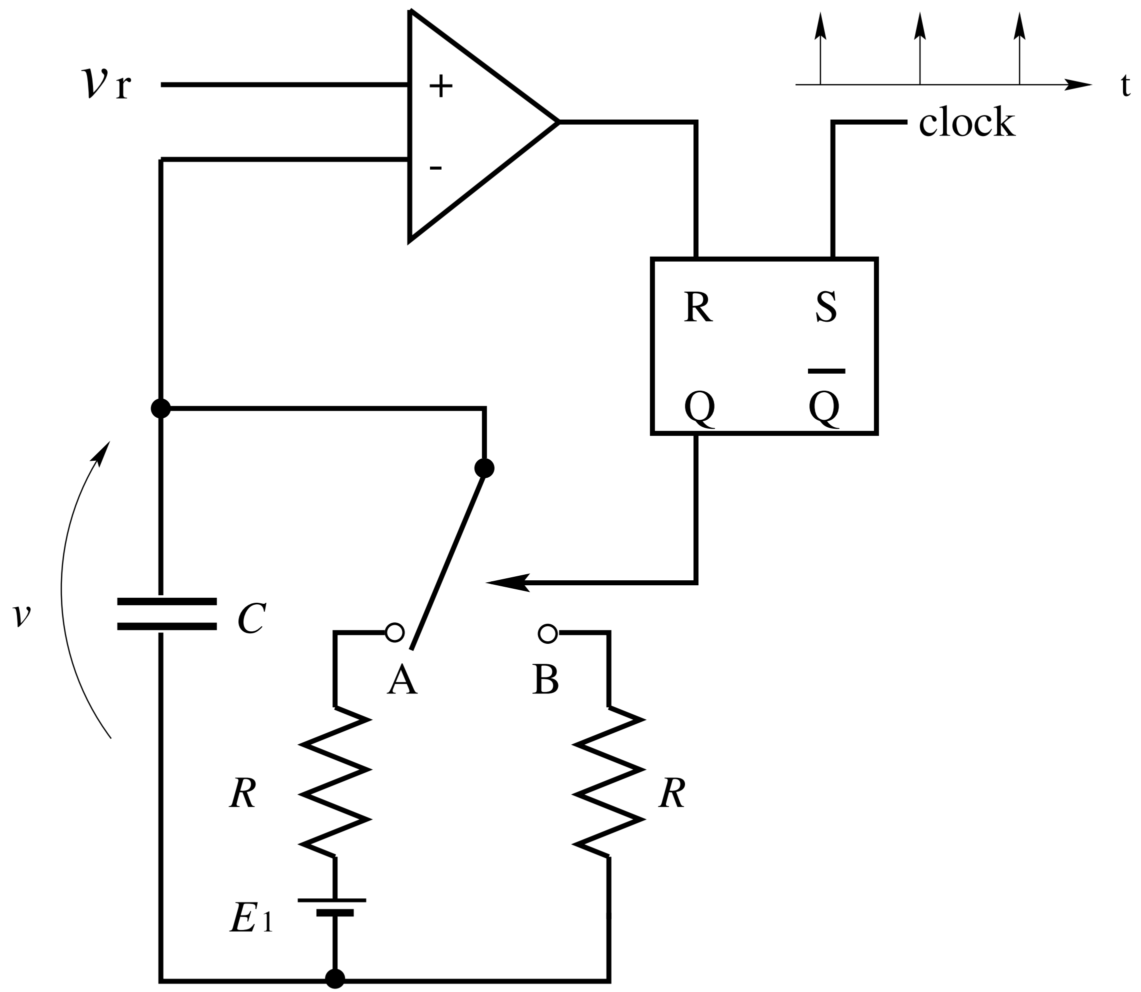
不安定周期解が高精度に求まることが恩恵

Border-Collision 分岐

- ✎ 離散写像の特性 $x_{k+1} = f(x_k)$ について，df/dx が連続でないときに生じる可能性
- ✎ 連続系でも起こり得る

局所的にテント写像と同じ構造になる

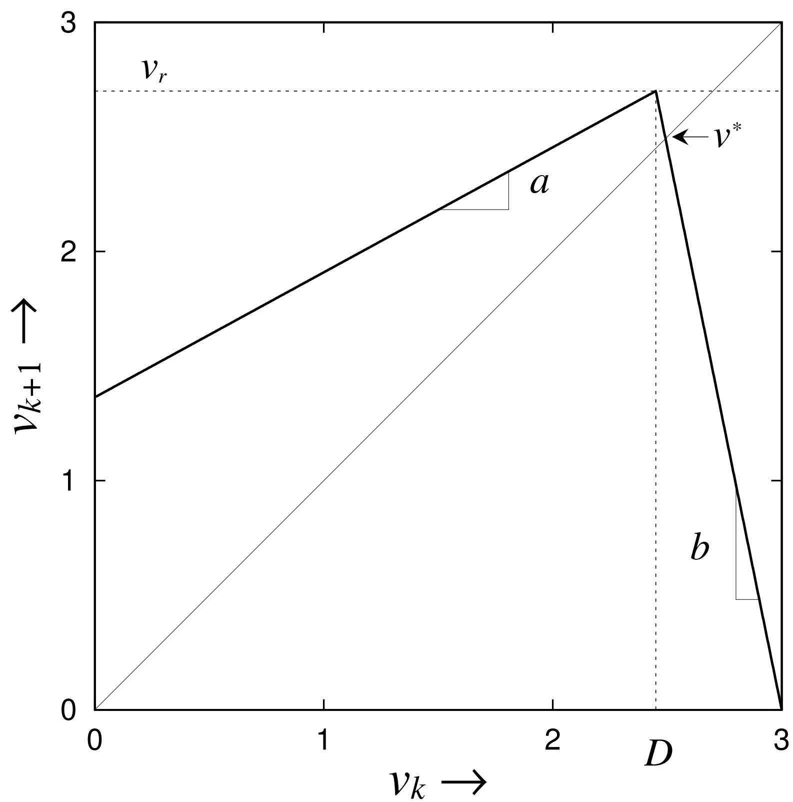
カオスを生じるもっとも簡単な回路



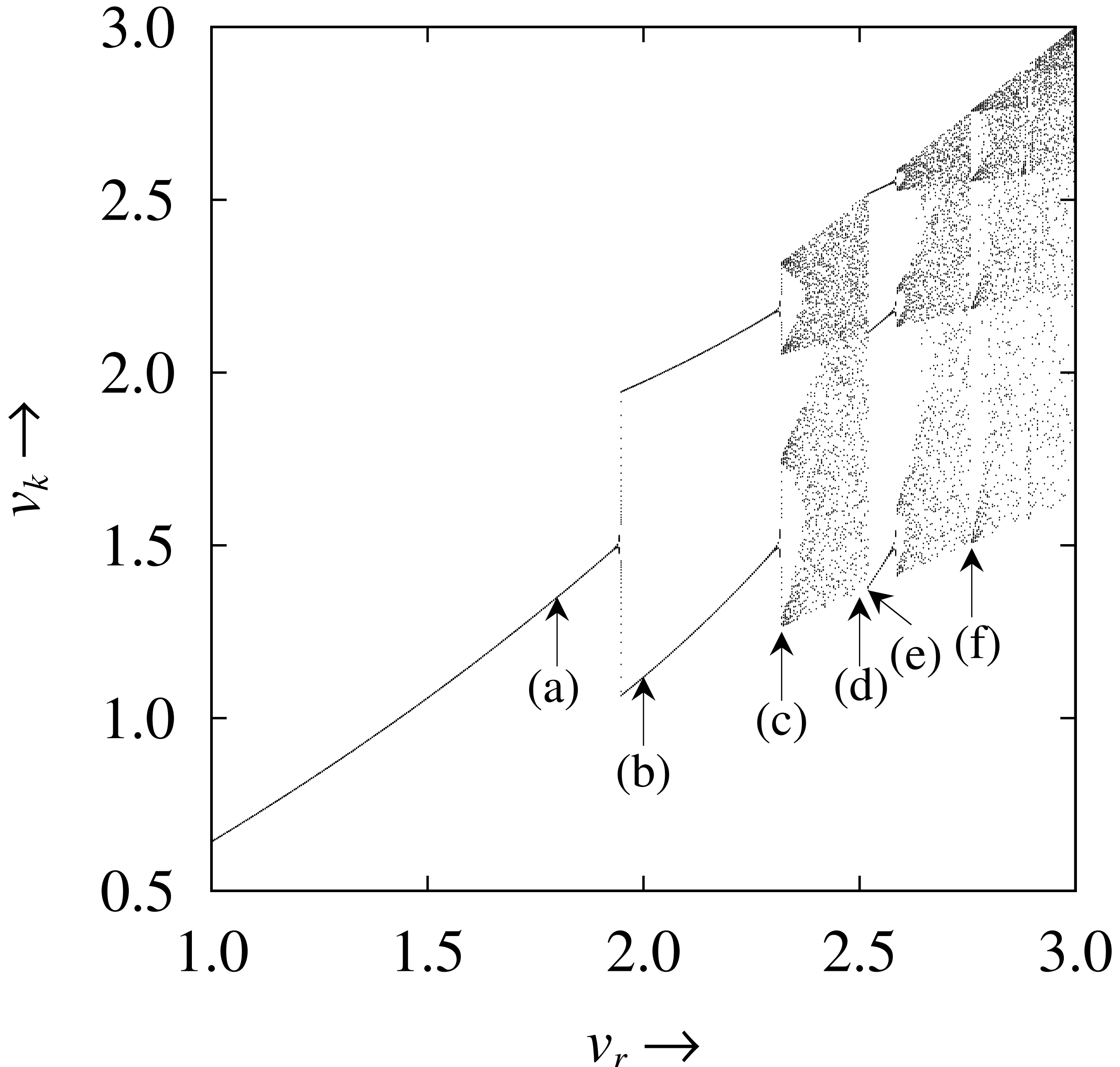
リターンマップ

$$v_{k+1} = \begin{cases} (v_k - E_1)e^{-T} + E_1 & \textbf{if} \quad v_k \le D \\ v_r \dfrac{v_k - E_1}{v_r - E_1} e^{-T} & \textbf{if} \quad v_k > D \end{cases}$$

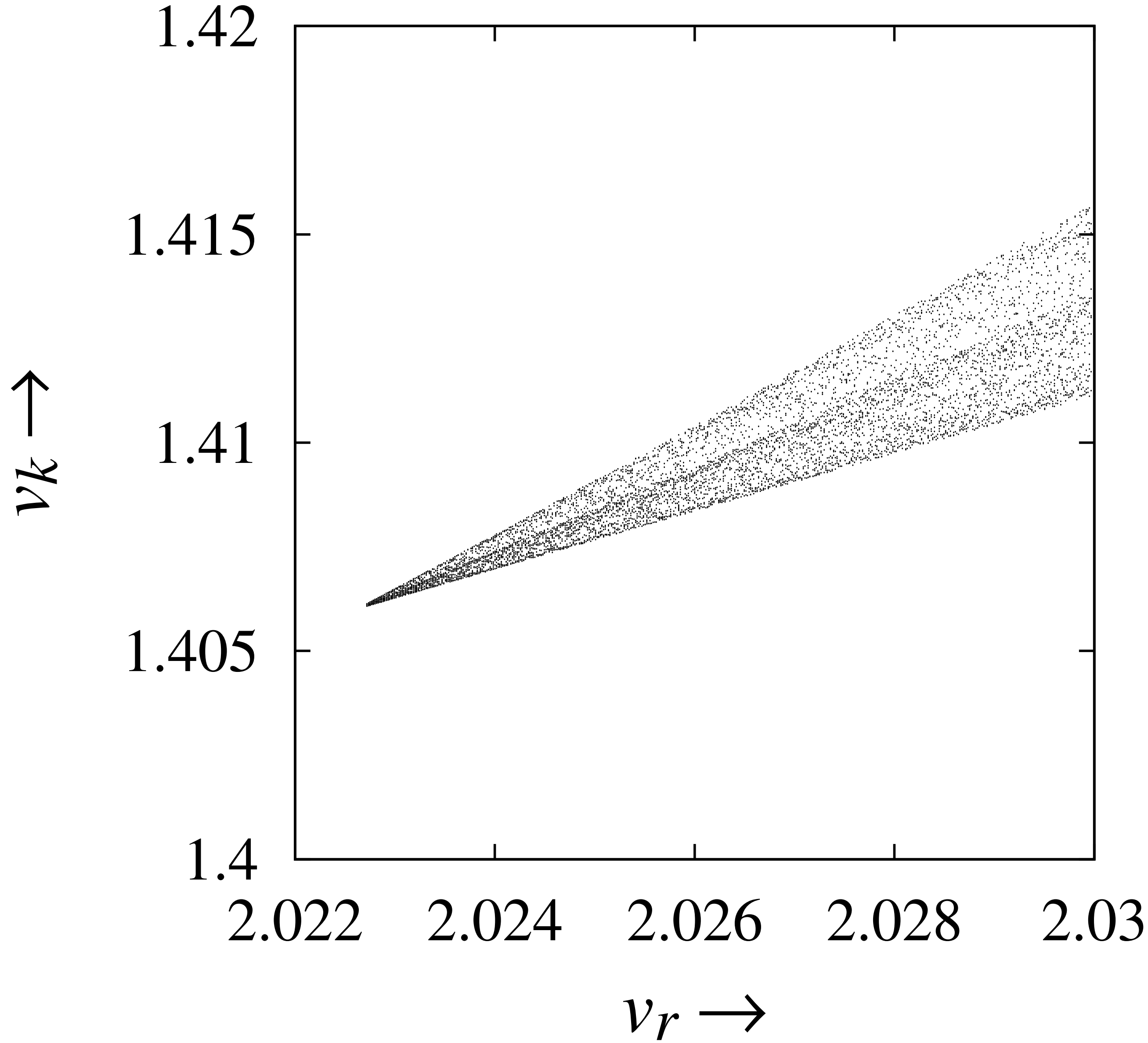
Border: $D = (v_r - E_1)e^{T} + E_1$



Bifurcation diagram

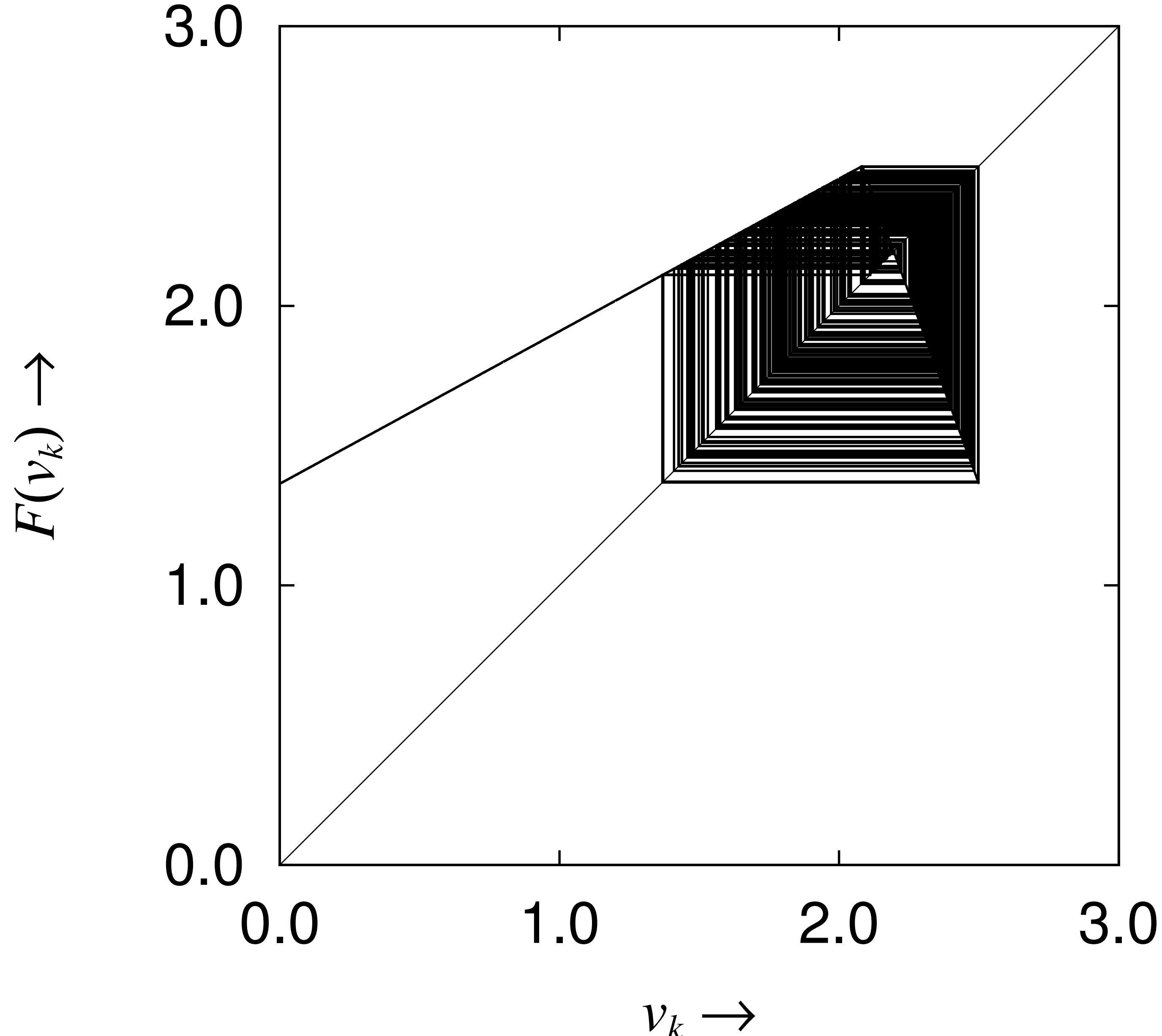


拡大図

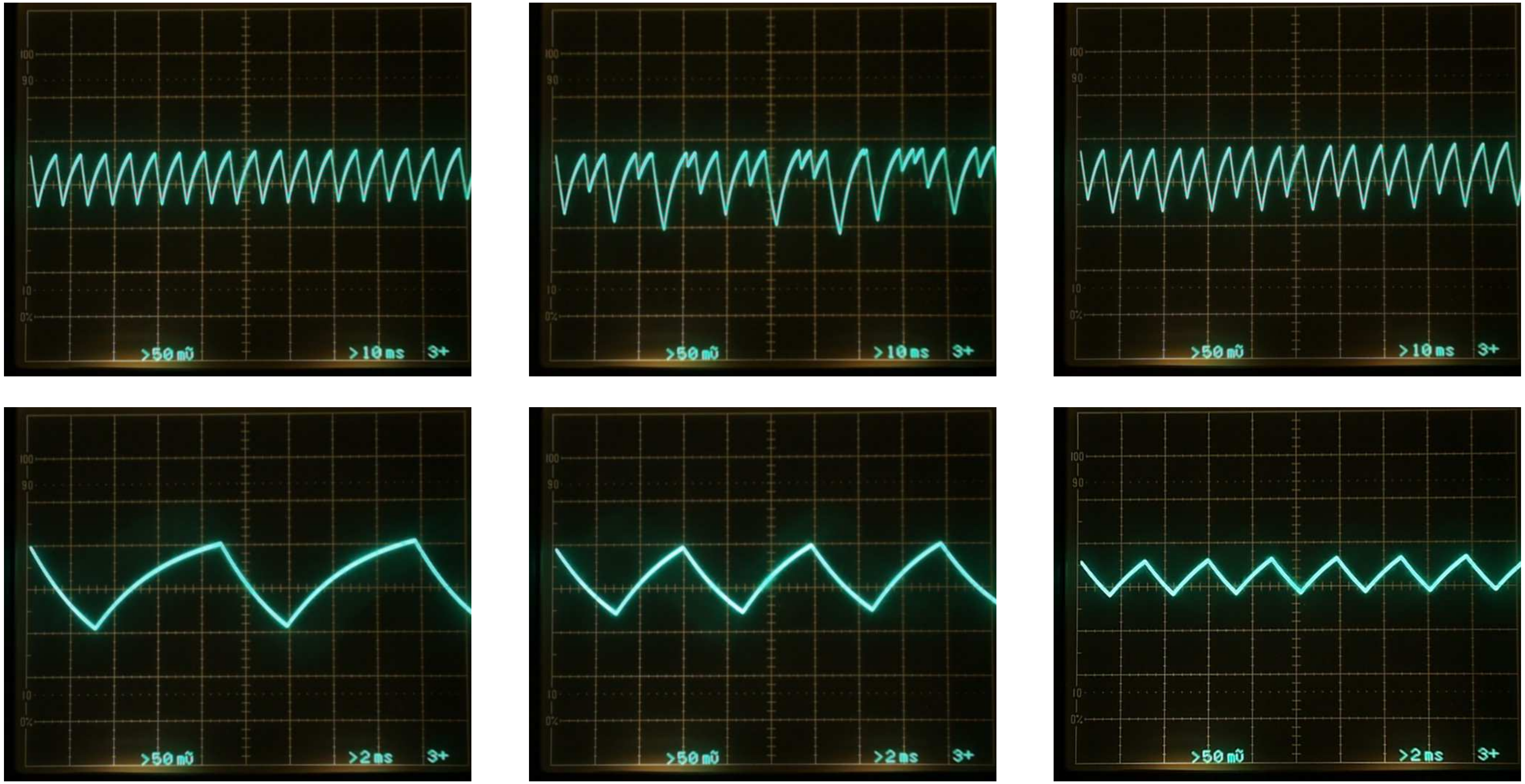


1.42
1.415
1.41
1.405
1.4
2.022
2.024
2.026
2.028
2.03
$v_k \rightarrow$
$v_r \rightarrow$

Trajectory

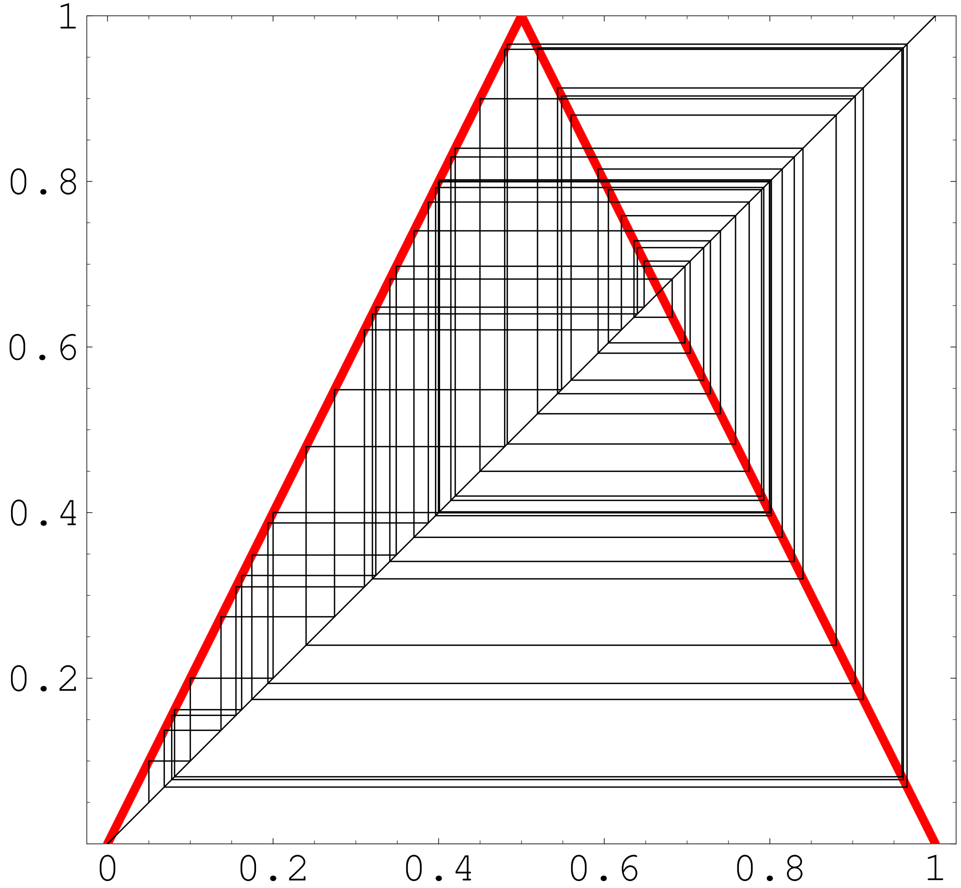


実験



Tent map

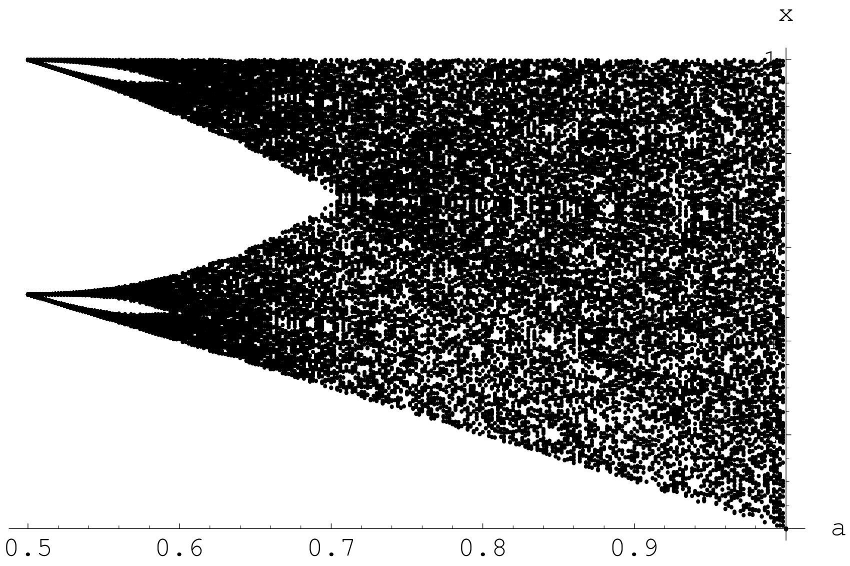
$$x_{k+1} = -\left|2a\left(x - \frac{1}{2}\right)\right| + 1$$



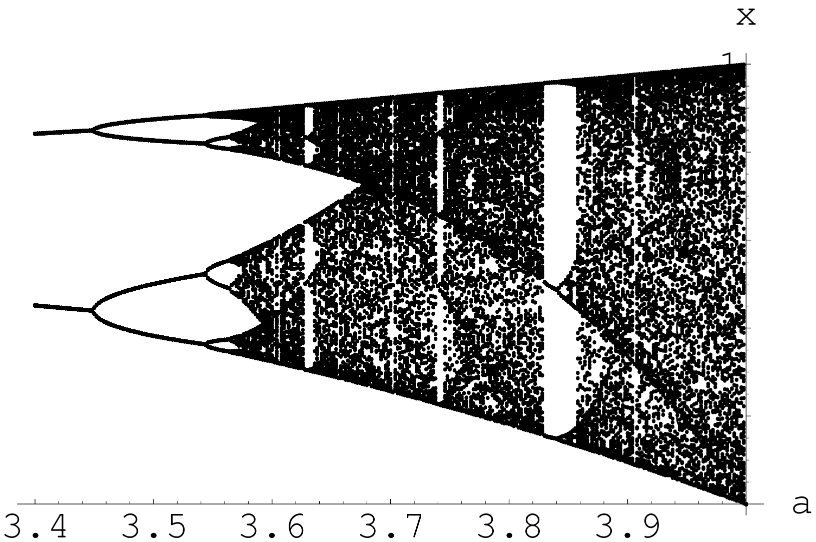
パラメータ a で分岐

Bifurcation diagram

Tent map:



Logistic map



BC分岐

- 滑らかでないことが原因となる分岐
- 広く実用コンバータ・インバータでも起こり得る
  - Prof. C. K. Tse (Hong Kong)
  - Prof. S. Banerjee (India)
- 応用: 一様なカオスを容易に生成することが可能?
  EMI に有効?

まとめ

ハイブリッドシステムの非線形力学系としての展望

- ✎ 分類
- ✎ **Poincaré 写像と周期解の計算**
- ✎ 応用: カオス制御など
- ✎ **BC 分岐**